Application of Finslerian Geometry to Raypaths in Complex Media

Andrej Bóna, Department of Mathematics and Statistics, University of Calgary, Alberta, Canada T2N 1N4
Michael A. Slawinski, Department of Earth Sciences, Memorial University of Newfoundland, ER4063 -- Alexander Murray Building, St. John's, NF, Canada, A1B 3X5

Summary

In complex media, raypaths and their associated traveltimes can be calculated by parameterizing the raypath parameter and using the principle of stationary traveltime (e.g., Epstein and Niatycki, 1992; Hanyga, 1996). A raypath parameter is a conserved quantity resulting from the symmetry of the velocity field. In an anisotropic, nonuniform[1] continuum, each and every point within this field can be described by an elementary wavefront - a closed curve in the infinitesimal neighbourhood of a virtual point source. A function that describes all elementary wavefronts is given by a metric of the associated geometry. For instance, the case where all elementary wavefronts of varying sizes, shapes and orientations are elliptical is described by a Riemannian metric. For more complicated elementary wavefronts, one uses a generalization of Riemannian geometry, namely, Finslerian geometry. A set of all elementary wavefronts defines the velocity field within which one computes the raypaths.

Introduction

Raypaths and their associated traveltimes in anisotropic, nonuniform media can be calculated by considering the stationarity of traveltime and a symmetry of the velocity field (e.g., Epstein and Slawinski, 1999). The calculus of variations (e.g., Slawinski and Webster, 1999) when combined with differential geometry (e.g., Ingarden, 1996), allows one to achieve such a formulation. The traveltime between the source and the receiver is given by the stationarity of a definite integral whose integrand is the ratio of a distance element and the velocity along this element, namely,

$$\delta \int_{A}^{B} \frac{ds}{V} = 0 \equiv \delta \int_{t_A}^{t_B} F dt \,, \qquad (1)$$

where t is the parameter along the raypath and F denotes the integrand of the traveltime integral.

Theory and method

In two-dimensional media, illustrated by an xz-plane, the raypath that results in a stationary traveltime must satisfy two Euler equations in terms of $x(t)$ and $z(t)$, namely

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial\left(\frac{dx}{dt}\right)} = 0 \,, \qquad (2)$$

and

$$\frac{\partial F}{\partial z} - \frac{d}{dt}\frac{\partial F}{\partial\left(\frac{dz}{dt}\right)} = 0 \,, \qquad (3)$$

[1] In this presentation, the term (non)uniformity is used to describe (in)homogeneity of a physical property of a medium since the latter term is used to characterize a mathematical property of a function. Also, such a nomenclature is correct in the context of continuum mechanics (e.g., Epstein and Slawinski, 1998).

(e.g., Morse and Feshbach , 1953). In general, considering perfectly elastic, anisotropic, nonuniform media, the traveltime integrand is a function of five arguments, namely, $F(x,z,\mathrm{d}x/\mathrm{d}t,\mathrm{d}z/\mathrm{d}t;t)$. In horizontally layered media, the velocity function is invariant to the horizontal translation, and the traveltime integrand becomes a function of four arguments, namely, $F(z,\mathrm{d}x/\mathrm{d}t,\mathrm{d}z/\mathrm{d}t;t)$. In accordance with Noether's theorem, this symmetry results in a conserved quantity (e.g., Goldstein, 1950), which, in geometrical optics, constitutes the raypath parameter (e.g., Kravtsov and Orlov, 1990), and, in the calculus of variations, is the first integral of the Euler equation (2), given by

$$p = \frac{\partial F}{\partial\left(\frac{dx}{dt}\right)}. \tag{4}$$

Consequently, this second-order differential equation (2) becomes a first-order equation (4).

An anisotropic, nonuniform medium can be described by associating its every point with an elementary wavefront (e.g., Arnold, 1989). In other words, at each point, there is a tangent space whose local coordinates depend on direction only. This allows one to account for the nonuniformity and anisotropy of the continuum while allowing one to view the medium, in the infinitesimal neighbourhood of every point, as locally uniform. Thus, the velocity of a signal at a given point depends on the propagation direction only. Hence, the velocity function is homogeneous of degree zero in $\mathrm{d}x/\mathrm{d}t$ and $\mathrm{d}z/\mathrm{d}t$, which results in the traveltime integrand being absolutely homogeneous of degree one in the same arguments, namely,

$$F\left(z, k\frac{dx}{dt}, k\frac{dz}{dt}; t\right) = |k| F\left(z, \frac{dx}{dt}, \frac{dz}{dt}; t\right). \tag{5}$$

From Euler's homogeneous-function theorem, it follows that

$$F = \frac{\partial F}{\partial\left(\frac{dx}{dt}\right)}\left(\frac{dx}{dt}\right) + \frac{\partial F}{\partial\left(\frac{dz}{dt}\right)}\left(\frac{dz}{dt}\right). \tag{6}$$

Also, the Euler equations (2) and (3) imply the Beltrami identity

$$\frac{\partial F}{\partial t} + \frac{d}{dt}\left(\frac{\partial F}{\partial\left(\frac{dx}{dt}\right)}\left(\frac{dx}{dt}\right) + \frac{\partial F}{\partial\left(\frac{dz}{dt}\right)}\left(\frac{dz}{dt}\right) - F\right) = 0. \tag{7}$$

Thus, due to the homogeneity of the function and in view of expression (6), the Euler identity (7) implies that the integrand cannot explicitly depend on the parameter t; hence, it becomes a function of three variables, namely, $F(z,\mathrm{d}x/\mathrm{d}t,\mathrm{d}z/\mathrm{d}t)$. Also, due to the homogeneity, and since $\tan\theta = \mathrm{d}x/\mathrm{d}z$, where θ is the ray angle, one can write the integrand as a function of two independent variables, namely, $F(z,\tan\theta,1)$. Furthermore, for perfectly elastic media, the Beltrami identity (7) combined with Euler's homogeneous-function theorem (6) implies that the Euler equation (3) is identically satisfied. As a result, and in view of the symmetry along x, the system of equations (2) and (3) is reduced to a single equation (4).

Consequently, the raypath parameter is a level curve of the surface spanned by z and $\tan\theta$. The raypath parameter can be solved for $\tan\theta$. Then, one can integrate this solution to get a raypath $[x,z]$. Subsequently, one can compute the traveltime by integrating F along this raypath between the source and the receiver.

Example

The properties of a medium allow one to determine the elementary wavefront, which is the indicatrix of the associated geometry. For elliptically anisotropic media such a geometry is provided by a Riemannian metric. Let the traveltime integral be

$$\int_{t_A}^{t_B} \sqrt{g_{xx}(z)\left(\frac{dx}{dt}\right)^2 + 2g_{xz}(z)\left(\frac{dx}{dt}\right)\left(\frac{dz}{dt}\right) + g_{zz}(z)\left(\frac{dz}{dt}\right)^2}\, dt, \tag{8}$$

where, as indicated by metric coefficients, $g_{ij}(z)$, the properties of the medium do not change along the *x*-axis. However, any angular or positional dependence along the *z*-axis can be accommodated. In view of the symmetry along the *x*-axis, one has a raypath parameter (4). Letting *F* be the integrand of the traveltime integral (8), one obtains

$$p = \operatorname{sgn}\left(\frac{dz}{dt}\right)\frac{g_{xx}(z)\tan\theta + g_{xz}(z)}{\sqrt{g_{xx}(z)\tan^2\theta + 2g_{xz}(z)\tan\theta + g_{zz}(z)}}. \tag{9}$$

Thus, the raypath parameter, *p*, is the level curve on surface spanned by *z* and $\tan\theta$. Rearranging expression (9), one obtains a quadratic expression for $\tan\theta$, namely,

$$g_{xx}\left(p^2 - g_{xx}\right)\tan^2\theta + 2g_{xz}\left(p^2 - g_{xx}\right)\tan\theta + p^2 g_{zz} - g_{xz}^2 = 0. \tag{10}$$

Equation (10) can be solved for $\tan\theta$. Subsequently, integration of this solution gives the raypath.

In a particular case of isotropy, where $g_{xz} = 0$, while $g_{xx} = g_{zz} = 1/V^2$, elementary wavefronts are indicatrices of Euclidean geometry. In this case, the integration of solution of equation (10) results in a familiar expression

$$x = \int_{x_1}^{x_2} \tan\theta dz = \int_{x_1}^{x_2} \frac{pV}{\sqrt{1 - p^2V^2}} dz. \tag{11}$$

In a more general case, where elementary wavefronts are complicated closed curves, one can invoke the Finslerian manifold where the angular velocity dependence is described by a locally Minkowskian tangent space.

Acknowledgments

The authors acknowledge PanCanadian Petroleum and Talisman Energy for their support of The Geomechanics Project.

References

Arnold, V.I., (1989) Mathematical methods of classical mechanics (2nd edition): Springer-Verlag

Epstein, M., and ► niatycki, J. (1992) ''Fermat's principle in elastodynamics'': Journal of Elasticity, 27, 45 - 56

Epstein, M., and Slawinski, M.A (1998) On some aspects of the continuum-mechanics context. Revue de l'Institut Français du Pétrole, 53, No. 5, pp. 669 - 677.

Epstein, M., and Slawinski, M.A. (1999) On rays and ray parameters in inhomogeneous isotropic media: Can. J. Expl. Geophys. 35 No. 1/2, pp. 7 - 19.

Goldstein, H., (1980) Classical mechanics: Addison-Wesley Publishing Co.

Hanyga, A., (1996) ''Fermat's principle and linearized tomography for anisotropic elasticity'': Proceedings of 7th SEG-IWSA

Ingarden, R., (1996) in Finsler geometry edited by Bao, D., Chern, S.-S., and Shen, Z.: American Mathematical Society

Kravtsov, Y.A., and Orlov, Y.I., (1990) Geometrical optics of inhomogeneous media: Springer-Verlag

Morse P.M., and Feshbach H., (1953) Methods of theoretical physics: McGraw-Hill, Inc.

Slawinski, M.A., and Webster, P.S. (1999) On generalized ray parameters for vertically inhomogeneous and anisotropic media: Can. J. Expl. Geophys. 35, No. 1/2, pp. 28 - 31.