Modeling kinematics for heterogeneous and anisotropic medium segments

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Summary

Kinematics and dynamics of seismic disturbance propagation are intimately linked, but for probing of fundamentals we may deal with one or the other separately. In this *abstract* and in the poster presentation I address two issues: (1) what are fundamentals re modeling the kinematics in *heterogeneous isotropic* and in *homogeneous anisotropic* medium segments separately, and also jointly as components within overall media; (2) re anisotropy I show that the dispersion relationship so central to the kinematics needs to be complemented by two additional constraints. From all this it manifests that along-energy-path disturbance propagation involves (long wavelength regime) Snell's law conforming paths with pathtime gradient detail different from front-normal slowness.

Motivational issues

Heterogeneity and anisotropy are pervasive in the earth, and phenomena stemming from these contribute to kinematics and dynamics detail in exploration geophysics probing and in seismology data. To what extent can we replicate such effects by composited medium models, i.e. in terms of combinations of segment models that each encapsulate the characteristics of concern, together with the spatial boundaries//segment interfaces within which they apply? The question so framed represents a problem//challenge in the realm of systems theory. A basic premise of systems theory pertaining to 'components//segments//phenomena in cascade configurations' is that accumulation variables relevant for the system overall and its model, must pertain also to components//segments//phenomena and their models, else how would we obtain the overall accumulations. A further basic premise is that continuity type phenomena or so-called *through*-variables be segment-internal consistent//continuous, also through-segment-boundaries//interfaces consistent as relevant for the pertaining phenomena, and to be also through-medium-total consistent. The component//segment characteristics must then be encapsulated as analytical representation models in terms of the applicable accumulation and continuity variables.

Seismic propagation, from some disturbance source, through composited medium segments of diverse character (with models as appropriate), to disturbance intercepting sensors, is essentially well understood, but not easily well modeled. Crudely summarized: (1) within segments, the disturbance propagates as a tangible energy wavefront, with energy-path//ray-path direction changes influenced by pointwise local heterogeneity, with progressive energy density changes, attenuation, etc.; (2) at interfaces, fronts can have energy multi-furcations (reflection/transmission/refraction/mode-conversions, …), and direction discontinuities, thereby creating a suite of new fronts of significant or lesser energy content, which makes some of these fronts prominent, others insignificant; (3) at interfaces also, bounding segment medium attributes, together with 'into'- and 'away-from'-interface directions of the fronts, contract to reflection//transmission coefficients, these then being convolution-encapsulated in dynamics-propagation variable amplitudes//spectral detail in the wavefront variables; (4) at arrayed sensors (near interfaces) the medium-internal boundary transitions convert to event-manifestions with pathtime-total_tags//sensor_location_tags, and with response signal amplitudes and response waveform detail (spectral content) from medium path total modulations, but presumably with dominance from a strong interface-event encapsulation.

For a systems context re kinematics, candidates for continuity phenomena//variables might be energy density flows and balances, but in ray theory valid situations obviously also ray direction unit vectors: they are continuous and Snell's law conformig re velocity fields within medium segments, and they admit discrete Snell's law consistent direction changes and multi-furcations at segment interfaces.

Clearly *pathtime* and *pathlength* are appropriate accumulations for paths through segments and for medium in composite. These alone suffice only for encapsuling a constant velocity field within a homogeneous isotropic single-segment medium. There must be other significant accumulations if essential velocity field detail for heterogeneous segments is be captured, or if constant velocity segments are combined, etc., and further, if cumulative such effects are expected to surface from aggregation totals of the appropriate variables. I have shown previously that for media comprised of heterogeneous isotropic lossless segments, for probing signal spectral content and medium segment scales where Snell's law paths are credible, triplets of path-specific {*pathtime, pathque*, *pathlength*} accumulations do actually encapsulate essentials which shape the detail in event wavefronts. What we can discern from seismic data is event manifestations that stem from energy furcations at path transitions through significant interfaces, call them event encapsulations, at a-priori unknown locations and at unknown pathtimes accumulated up to encapsulation. *Pathque* will be defined below (new variable, verbalizes representation letter q and points to context, is better name than previous *qvel* in Vetter, 2001).

Re kinematics, sensor data from partial intercept of wavefronts reveal only event manifestations and path-specific total pathtimes. Implicit, and linking to the observables, are geometic//spatial path detail and velocity field detail, interfaces, pointwise along-path direction and propagation speeds, progressive accumulation of variable totals, also implicitly the through-segment totals, among others. Through good understanding of the phenomena and with much effort we succeed to decipher some of the significant propagation detail, and to infer therefrom significant information about the medium probed.

Use designation l_{AC} for *pathlength* along a Snell's law conforming path between points A and C, the line integral accumulations of path infinitessimals; next t_{AC} for *pathtime*, the path-associated pointwise-slowness-weighted integrated length; and also q_{AC} for *pathque*, the path-associated pointwise-velocity-weighted integrated length. In a constant velocity medium it is the simple product $q_{AC} = l_{AC}\, v$. The significance of these path-specific accumulation variables stems from their link to means of the associated path-specific *slowness density* (or occurrence frequency) and *velocity density* distributions, by relationships below (Vetter 2001):

t_{AC} / l_{AC} = mean of along-path slownes density distribution (= path mean slowness)
l_{AC} / t_{AC} = harmonic mean of along-path velocity density distribution (= time average velocity)
q_{AC} / l_{AC} = mean of along-path velocity density distribution (= path mean velocity)
$(q_{AC} / t_{AC})^{1/2}$ = geometric mean of (mean and harmonic mean) of along-path velocity density distribution (= pathRMS velocity)

Pathtimes relevant to marker-events (e.g. pertaining to a common shot reflection event front cone intercepted by an areal sensor group), can be discerned from probing data. *Pathlength*s and *pathque*s are somewhat problematic, but if we can obtain somehow good estimates, these will be useful towards discerning certain particulars of the raypaths and medium segment interfaces, also of confidence bounds for their locations and orientations . *Pathque* encapsulates also the geometrical spreading essentials for segments and for segments cumulative along paths. Note that *path-RMS velocity* re along-path distributions differs from the parametric v_{RMS} used in the Dix method for interval velocity estimation. Those parametric v_{RMS} are effectively some average of *path-RMS velocities* over an ensemble of usually small offset paths. Conversely, when we have discerned a parametric v_{RMS} for paths to a sensor group (line or areal) for some event-front move-through, that v_{RMS} with *pathtimes* t_{AC} gives $q_{AC\text{-}aprx} = v_{RMS}{}^2 t_{AC}$ approximations .

As indicated earlier, certain propagation velocity//slowness detail along Snell's law paths in isotropic heterogeneous media is encapsulated in $\{t_{AC}, q_{AC}, l_{AC}\}$ variable triplets, and that detail encrypts into wavefront detail. The wavefront when intercepted by sensors yields 'moveout ', or for the 2D sensing context 'front movethrough' with path-specific directionnally sensed *moveout velocity* manifestation. All this is compelling motivation to ponder and explore whether there should not also be Snell's law paths and $\{t, q, l\}$ accumulations involved in disturbance propagation through anisotropic medium segments, for long wavelength regimes where we can deem anisotropy models to apply. Is anisotropy not essentially just encapsulation somehow of small scale ordered heterogeneity?

Frame this as a CONJECTURE: Snell's law compatible paths associated with, but distinct from, the energy paths exist for the long-wavelenth-relevant homogeneous anisotropic medium representation models. If we discern the paths and along-path velocities, we can expect to discern also *pathlength* and *pathque* accumulations. With such understanding analytically encapsulated, we would have a sound basis for conjoining segment models for the different attributes. We could be confident that the medium overal kinematics are appropriately modeled and, when wavefront movethrough can be discerned, that the attendant moveout velocities do link to *path-RMS velocities* re the velocity//slowness distributions. There is more which is particularly relevant to from-data-to-model inversing.

But what about our understanding that Snell's law paths are exact only for infinitely short wavelength (c.f. Helbig 1994,p6), presumably precluding therefore the codicil of relevance for propagation phenomena in real anisotropic media? This is a concern to ponder, but it need not compel us to abandon exploring the conjecture. The argument can be made that, for ideal anisotropy in the sense that the scale of local structure in the medium segment goes to the infinitessimal, even the very high frequency spectral components of a disturbance translate to the long wavelength regime. If we can clarify the issue for this 'scale-extremum', then we can expect relevance of the concept also for propagation of real disturbances in real anisotropic medium segments, subject to the long wavelenth regime condition, viz. that wavelength/scale ratio is sufficiently large, 10 or greater generally deemed sufficient. The issue of wavelengths and ordered structure scale is indeed very important (Helbig 1974, Thomsen 2002, many others) .

The Snell's law paths

Pierre de Fermat (1657) has pronounced his perceptive insights that "the path of a ray of light between two points is that particular one which minimizes the traveltime". Fermat's principle broader applicability encompasses a statement like "between a shot and a sensor, a seismic disturbance progresses along traveltime minimizing paths (spectral components specific) " . Can we then doubt that wavefronts evolve as they must, or here for homogeneous (lossless, infinitessimal scale) anisotropic medium relevance, that the straight-line-idealized energy paths are the pathtime minimizing **energy paths**? Of course not! Nor have we reason to doubt, however, that parameters for wavefront-normal direction and wavefront-normal velocity magnitude, associated with the energy paths, have values that honour the pathtime minimizing conditions. Can we get more transparency on the link between front-normal directed velocity and energy path velocity? And is the path of energy propagation in seismics the analog of *the path of a ray of light* ?

For each or the propagation modes, front-normal velocity magnitude, by virtue of the Kelvin-Christoffel matrix and the familiar dispersion relationship, beyond dependence on medium density and elastic parameters, is functionally dependent on the relevant front-normal direction parameters. Deem observation coordinate frame coincident with anisotropy-natural frame, and visualize a shock front commencing from the origin. Whether we search for Snell's law paths or not, every particular energy path has associated front-normal angle parameters $\{\theta, \phi\}$ with values consistent with pathtime minimizing conditions re these parameters. Analytical expressions for these conditions will encapsulate relevant constraints. The motivation and reasoning outlined leads us to novel compact relationships that link energy path vector velocity $\boldsymbol{v}_E$ to front-normal velocity magnitude v_N plus its derivatives wrt the front-normal direction angles. A concise vector form, in particular, is very transparent and facilitates visualization (Appendix).

Still, how can we reconcile pathtime acccumulation and pathtime gradient (**deemed to be** front-normal slowness vector) with along-path conceptual direction-preserving jump-discontinuities of the gradient incrementals? Rather, might pathtime gradient not have medium-internal along-path continuity with direction and slowness-magnitude discontinuities? How to clarify the issue (Vetter 1993,1999a 1999b)? Represent pathtime as an along-path line-integral accumulation of pointwise pathtime gradient. Next, examine all path directions with their velocities that accord with the front-normal slowness components in all possible combinations. Do this by integrating over path incrementals, then later shrink these to infinitesimals. Front-normal direction manifests as feasible, but so do all directions in medium-natural coordinate planes where just two of the front-normal slowness components participate, and along coordinate axes where just one of them participates. Seven possible combinations of front-normal slowness components, seven categorical directions (unit vectors) with their categorical velocities. The expressions are combinations of front-normal velocity magnirtude and directions, $V_{PATH\text{-}SEGS} = \text{fn}\ \{v_N , \theta, \phi\}$ (Appendix); they do not therefore conflict with dispersion relationship dictates.

And with all the above discerned it is not difficult to put the pieces together. Aiming to visualize the pathtime gradient detail and to determine *pathtime-*, *pathlength-*, and *pathque* totals within medium segments, I aggregate infinitesimal same-direction incrementals; the result is three-segment gradient representation models (or two-, or only one- when front-normal and energy path direction coincide). There are six 3D-anisotropy path categories (Appendix), also special cases for 2D-polar-anisotropy. Applicability depends on the conjunction of front-normal and the energy path velocities with their directions. Visualizing shot and energy front arrival points as diagonally opposite corners of a rectanguloid box, the gradient representation path is along the front-normal, then in a plane, and lastly along an edge of the box. Direction changes between the segments are Snell's law conformiing; we can visualize them re the 'box' as *akin to body wave modes* with at-interface headwave refractions into and from adjacent velocity regimes. Conjoined gradient directions are therefore *rays* in the postulated sense of ray theory and Fermat's principle. And clearly also, the energy propagation paths are not the analogs of *the paths of a ray of light*. And the **so-called rays** along energy path directions **are not-rays**.

Conclusions

Re kinematics in complex media broadly, I have pointed out that explicit//implicit {*pathtime, pathque, pathlength*} accumulations are essential for linking event-wavefront detail to by-path-traversed velocity field detail. Among others, along-path-v_{RMS} manifests as an encapsulation within the variable triplet. Re anisotropy kinematics, the analytical relationships detailed 'explain' the puzzling complexities of velocities and wavefronts. Long wavelength regime kinematics are ray-theory consistent and tractable. This has implications, viz. for kinematics broadly incl. geometrical spreading, for raypath tracing through composited media, potentially also for inversing discerned kinematics phenomena to confident estimates of geometrical and medium parametric detail, and more .

References

Auld,B.A.,1973, Acoustic Fields and Waves in Solids, vol I, John Wiley and Sons, New York

Helbig,K.,1994, Foundations of Anisotropy for Exploration Seismics, Pergamon

Mensch,T., Rasolofosaon,P.,1997, Elastic-wave velocities in anisotropic media …– generalization of Thomsen's epsilon, delta, gamma: Geophys. J. Int. 128, 43-64.

Rudzki, M.P.,1911, Parametrische Darstellung der elastischen Wellen in anisotropen Medien, Bull. Acad. Cracovie 503-536
(Parametric representation of elastic waves/ wavefronts in anisitropic media)

Thomsen, L , 2002, Understanding Seismic Anisotropy in Exploration and Exploitation, SEG/ EAGE Short Course Series No.5,

Vetter,W.J.,1993, The Fermat path for ray and wavefront kinematics in anisotropic media, 63 rd Ann. Intn. Mtg Soc. Expl. Geophys., Abstr. 953-956

Vetter,W.J.,1997, Wavefronts and velocities in uniaxially anisotropic media, 67th Ann. Intn. Mtg Soc. Expl. Geophys., Abstr.1533-1536

Vetter,W.J.,1999a,Kinematics in a 3D anisotropic medium: "Snell's law path" and its canonical model, Canadian Soc. Expl. Geoph., Techn. Abstr. 129-132

Vetter,W.J.,1999b,Kinematics in a 3D anisotropic medium: Snell's law paths and their canonical model, 67th Ann. Intn. Mtg Soc. Expl. Geophys., Abstr.1887-1891

Vetter,W.J.,2001, Pathlength, pathtime, and "qvel": the essential variables for along-raypath aggregated velocities and heterogeneity,
Canadian Soc.Expl. Geophys. CD Techn. Abstr. 'The Gifts of Promeththeus (Processing) 16-18

Appendix: principal analytical relationships

Readers will have some understanding of elasticity concepts that lead to encapsulation of elastic parameters with the propagation front-normal unit direction vectors, pertaining to propagation of a plane-wave composited disturbance from-point-shot to energy front points, and to the path-specific energy-path-linked 3x3-Kelvin-Christoffel matrix $\Gamma(c_{ij}, n_x, n_y, n_z)$. The eigenvalues of $\Gamma(\ldots)$ are elasticity moduli $M = \rho(v_{N\,m})^2$, (index m = qP, qSV, qSH), that encapsulate mode-distinct front-normal velocities associated with the energy paths, and eigenvectors are polarization directions of the respective vibratory modes (Auld 1973, Helbig 1994, Mensch..1997, Thomsen 2002, others). For context here we need to use the familiar cubic polynominal expression for mode-relevant elasticity moduli $M = \rho(v_{N\,m})^2$ in terms of elastic parameter-, plus density-, plus front-normal direction unit-vector components explicitly ; likewise mode-relevant front-normal velocities in terms of . . . explicitly; all this because we need to compute values for v_N-derivative expressions re angle direction parameters {*theta , phi* }.

Equation (1), credited to Rudzki (1911; c.f. Helbig 1994 p.6), encapsulates the time-progressive wavefront from a long wavelength regime point disturbance at origin of medium-natural//observation coordinate frame to energy front points {*x, y, z*}. Front-normal velocity magnitude $v_{N\,m}$ depends on medium parameters and front-normal direction. Equation (1) yields pathtime minimum constraint conditions (2) and (3). By rearrangement of those three equations we obtain relationships (4) and then matrix inversed equivalent thereto, relationships (5). Summing squares of the left-hand side energy path velocity vector components in (5) yields the Pythagoras theorem revealing relationship (6).

The relational detail in (5) compacts to the simple and profoundly transparent vector equation (7). It reveals that and how directional derivatives of v_N(c_{ij} , *theta, phi*) contribute to composited energy path velocity magnitude and direction. We can encapsulate these details on front-normal velocity surface plots: plot three velocity dimensioned surfaces for front-normal directions, {v_N , v_N+dv_N/d*theta*, v_N+(dv_N/d*phi*)/sin*theta* }; the derivative terms {+/-}information hugs the velocity surface. With { $\vec{1}_N$, $\vec{1}_\theta$, $\vec{1}_\phi$ } implicit through front-normal directions, we can visualize (or locate computationally) linked energy front vector velocity as encapsulated in equations (5, 6, 7). Front-normal three-surface representations are visually informative and also 'quantitative indicators' of the directional severity of anisotropy of a given medium (inspect equations (6) and (7)!).

Re Snell's law representation paths, expressions for path *vector*, segmented *pathlength* and *pathque,* parallel those shown for case (1), eqn(8).

$$t\, v_N = x \sin\theta\cos\phi + y \sin\theta\sin\phi + z\cos\theta \qquad (1)$$

$$\frac{\partial t}{\partial\theta} = 0 = -\frac{1}{{v_N}^2}\frac{\partial v_N}{\partial\theta}(x\sin\theta\cos\phi + y\sin\theta\sin\phi + z\cos\theta) + \frac{1}{v_N}(-x\cos\phi\cos\theta + y\sin\phi\cos\theta - z\sin\theta) \qquad (2)$$

$$\frac{\partial t}{\partial\phi} = 0 = -\frac{1}{{v_N}^2}\frac{\partial v_N}{\partial\phi}(x\sin\theta\cos\phi + y\sin\theta\sin\phi + z\cos\theta) + \frac{1}{v_N}(-x\sin\phi\sin\theta + y\cos\phi\sin\theta) \qquad (3)$$

$$\begin{pmatrix} v_N \\ \dfrac{\partial v_N}{\partial \theta} \\ \dfrac{1}{\sin\theta}\dfrac{\partial v_N}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} \dfrac{x}{t} \\ \dfrac{y}{t} \\ \dfrac{z}{t} \end{pmatrix} \quad (4) \qquad \begin{pmatrix} \dfrac{x}{t} \\ \dfrac{y}{t} \\ \dfrac{z}{t} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} v_N \\ \dfrac{\partial v_N}{\partial \theta} \\ \dfrac{1}{\sin\theta}\dfrac{\partial v_N}{\partial \phi} \end{pmatrix} \quad (5)$$

$$\left(\frac{x}{t}\right)^2 + \left(\frac{y}{t}\right)^2 + \left(\frac{z}{t}\right)^2 = v_E^{\,2} = v_N^{\,2} + \left(\frac{\partial v_N}{\partial\theta}\right)^2 + \left(\frac{1}{\sin\theta}\frac{\partial v_N}{\partial\phi}\right)^2 = v_N^{\,2} + \left(\sqrt{\left(\frac{1}{\sin\theta}\frac{\partial v_N}{\partial\phi}\right)^2 + \left(\frac{\partial v_N}{\partial\theta}\right)^2}\right)^2 \quad (6)$$

$$\boldsymbol{r}_{AC} = t\boldsymbol{v}_E = t\,(\,v_N\,\vec{1}_N + \frac{\partial v_N}{\partial\theta}\,\vec{1}_\theta + \frac{1}{\sin\theta}\frac{\partial v_N}{\partial\phi}\,\vec{1}_\phi\,) \quad (7)$$

<u>Long wavelength regime 3D Snell's law representation paths (direction/velocity cosistent infinitessimals aggregated)</u>
<u>unit vectors:</u> $\{\mathbf{1}_x, \mathbf{1}_y, \mathbf{1}_z\}$ observation coordinate frame <u>aligned with</u> anisotropy structure-natural (or crystal-natural) frame;

$\mathbf{1}_N = \vec{n} = (\sin\theta\cos\phi)\,\mathbf{1}_x + (\sin\theta\sin\phi)\mathbf{1}_y + (\cos\theta)\mathbf{1}_z$	$V_N = v_N$	(in 3D general directions)
$\mathbf{1}_Z = (\cos\phi)\,\mathbf{1}_x + (\sin\phi)\mathbf{1}_y$	$V_Z = v_N/\sin\theta$	(in planes z = constant)
$\mathbf{1}_X = ((\tan\theta\sin\phi)\mathbf{1}_y + \mathbf{1}_z)/(1+\tan^2\theta\sin^2\phi)^{1/2}$	$V_X = v_N/\cos\theta(1+\tan^2\theta\sin^2\phi)^{1/2}$	(in planes x = constant)
$\mathbf{1}_Y = ((\tan\theta\cos\phi)\mathbf{1}_x + \mathbf{1}_z)/(1+\tan^2\theta\cos^2\phi)^{1/2}$	$V_Y = v_N/\cos\theta(1+\tan^2\theta\cos^2\phi)^{1/2}$	(in planes y = constant)
$\mathbf{1}_x$	$V_x = v_N/\sin\theta\cos\phi$	(along $\mathbf{1}_x$ directions)
$\mathbf{1}_y$	$V_y = v_N/\sin\theta\sin\phi$	(along $\mathbf{1}_y$ directions)
$\mathbf{1}_z$	$V_z = v_N/\cos\theta$	(along $\mathbf{1}_z$ directions)

(1) NZx : $(\underline{x_N < x_E},\ y_N > y_E,\ z_N > z_E)$; $y_E > z_E\sin\phi\tan\theta$; $x_E > y_E/\tan\phi$ (8a)
$\boldsymbol{r}_{AC} = \boldsymbol{l}_{NZ\,x} = (z_E/\cos\theta\,\mathbf{1}_N + (y_E/\sin\phi - z_E\tan\theta)\mathbf{1}_Z + (x_E - y_E/\tan\phi)\mathbf{1}_x$ (8b)

$l_{AC} = l_{NZ\,x} = (z_E/\cos\theta) + (_{FE}/\sin\phi - z_E\tan\theta) + (x_E - y_E/\tan\phi)$ (8c)

$t_{AC} = t_{NZ\,x} = (z_E/\cos\theta)//(v_N) + (y_E/\sin\phi - z_E\tan\theta)//(v_N/\sin\theta) + (x_E - y_E/\tan\phi)//(v_N/\sin\theta\cos\phi)$ (8d)

$q_{AC} = q_{NZ\,x} = (z_E/\cos\theta)*(v_N) + (y_E/\sin\phi - z_E\tan\theta)*(v_N/\sin\theta) + (x_E - y_E/\tan\phi)*(v_N/\sin\theta\cos\phi)$ (8e)

(2) NZy : $(x_N > x_E,\ \underline{y_N < y_E},\ z_N > z_E)$; $x_E > z_E\tan\theta\cos\phi$; $y_E > x_E\tan\phi$ (9a)
$t_{AC} = t_{NZ\,y} = (z_E/\cos\theta)//(v_N) + (x_E/\cos\phi - z_E\tan\theta)//(v_N/\sin\theta) + (y_E - x_E\tan\phi)//(v_N/\sin\theta\sin\phi)$ (9d)

(3) NXy : $(x_N > x_E,\ \underline{y_N < y_E},\ z_N > z_E)$; $z_E\tan\theta\cos\phi > x_E$; $y_E > z_E\tan\theta\sin\phi$ (10a)
$t_{AC} = t_{NX\,y} = (x_E/\sin\theta\cos\phi)//(v_N) + (z_E - x_E/\tan\theta\cos\phi)(1+\tan^2\theta\sin^2\phi)^{1/2}//(v_N/\cos\theta(1+\tan^2\theta\sin^2\phi)^{1/2})$
$+ (y_E - z_E\tan\theta\sin\phi)//(v_N/\sin\theta\sin\phi)$ (10d)

(4) NXz : $(x_N > x_E,\ y_N > y_E,\ \underline{z_N < z_E})$; $y_E > x_E\tan\phi$; $z_E\tan\theta\sin\phi > y_E$ (11a)
$t_{AC} = t_{NX\,z} = (x_E/\sin\theta\cos\phi)//(v_N)$
$+ ((y_F/\tan\theta\sin\phi - x_E/\tan\theta\cos\phi)(1+\tan^2\theta\sin^2\phi)^{1/2})//(v_N/\cos\theta(1+\tan^2\theta\sin^2\phi)^{1/2})$ (11d)
$+ (z_E - y_E/\tan\theta\sin\phi)//(v_N/\cos\theta)$

(5) NYx : $(x_N < x_E,\ y_N > y_E,\ z_N > z_E)$; $z_E\tan\theta\sin\phi > y_E$; $x_E > z_E\tan\theta\cos\phi$ (12a)
$t_{AC} = t_{NY\,x} = (y_E/\sin\theta\sin\phi)//(v_N) + ((z_E - y_E/\tan\theta\sin\phi)(1+\cos^2\phi\tan^2\theta)^{1/2})//(v_N/\cos\theta(1+\tan^2\theta\cos^2\phi)^{1/2})$
$+ (x_E - z_E\tan\theta\cos\phi)//(v_N/\sin\theta\cos\phi)$ (12d)

(6) NYz : $(x_N > x_E,\ y_N > y_E,\ \underline{z_N < z_E})$; $x_E\tan\phi > y_E$; $z_E\tan\theta\cos\phi > x_E$ (13a)
$t_{AC} = t_{NY\,x} = (y_E/\sin\theta\sin\phi)//(v_N)$
$+ ((x_E/\cos\phi\tan\theta - y_E/\sin\phi\tan\theta)(1+\cos^2\phi\tan^2\theta)^{1/2})//(v_N/\cos\theta(1+\tan^2\theta\cos^2\phi)^{1/2}$ (13d)
$+ (z_E - x_E/\tan\theta\cos\phi)//(v_N/\cos\theta)$