Kinematic and dynamic raytracing in anisotropic media - a phase-velocity formulation

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2005 CSEG National Convention

Abstract

Kinematic and dynamic raytracing in inhomogeneous, anisotropic media has been traditionally formulated in terms of elastic parameters. Such a formulation is complicated and inefficient for computation as it requires solving an eigenvalue problem at each ray step. It also requires that a medium be specified with elastic parameters. This is inconsistent with the common practice in seismic data processing where anisotropy is usually described with Thomsen parameters. This inconsistency may result in ambiguity in specifying the elastic parameters. To overcome these difficulties, we have reformulated the kinematic and dynamic raytracing systems in terms of phase velocity. The new formulation is much simpler and computationally more efficient than the previous elastic parameter based formulations since solution of the eigenvalue problem at each ray step is no longer required. The efficiency of the dynamic raytracing system is further enhanced by using a newly proposed nonorthogonal ray-centered coordinate system. Since the medium for raytracing is now specified with phase velocity, the possible ambiguity in specifying elastic parameters is also eliminated. The kinematic and dynamic raytracing systems developed in this study thus provide a useful and efficient tool for seismic modeling and imaging in anisotropic media, especially for the transversely isotropic and orthorhombic media where simple analytical expressions for phase velocity have been derived by Thomsen (1986) and Tsvankin (2001), and the integration of the ray tracing systems can be carried out relatively inexpensively using these expressions.

Introduction

Kinematic and dynamic raytracing in inhomogeneous, anisotropic media is an essential building block for seismic modeling and imaging with ray methods. Kinematic raytracing is required for traveltime modeling while dynamic raytracing is needed for ray amplitude calculation. The latter can also be used to construct Gaussian beams along a ray. Alkhalifah (1995), for example, has employed this technique for anisotropic Gaussian-beam depth migration. Kinematic raytracing in anisotropic media has traditionally been formulated in terms of elastic parameters (Cerveny, 1972; Kendall and Thomson, 1989). Such a formulation is, however, physically not intuitive and computationally cumbersome (Cerveny, 1989). Moreover, it requires a medium to be specified with elastic parameters. The common practice in seismic data processing, on the other hand, is to describe anisotropy with Thomsen parameters. This inconsistency may cause problems in medium specification. For example, it may result in ambiguity in specifying elastic parameters for P-wave imaging in weak transversely isotropic (TI) media. The elastic parameter based formulation for anisotropic dynamic raytracing is even more complicated as it now involves differentiation of kinematic ray equations with respect to ray parameters (Hanyga, 1986; Cerveny, 2001). The purpose of this study is to formulate the kinematic and dynamic raytracing systems in anisotropic media in terms of phase velocity. This formulation overcomes some of difficulties of the elastic parameter based formulation, and is especially useful for TI and orthorhombic media where simple analytic expressions for phase velocity have been derived in terms of the Thomsen parameters (Thomsen, 1986; Tsvankin, 2001).

Kinematic raytracing system

The kinematic raytracing equations in anisotropic media have been derived by Cerveny (1972); we summarize here only the results needed for this study. We start with the frequency-domain equation of motion in inhomogeneous, anisotropic media:

$$\frac{\partial}{\partial x_i}\left(c_{ijkl}\frac{\partial u_k}{\partial x_l}\right)+\omega^2\rho u_j=0, \qquad (1)$$

where u_i is displacement, c_{ijkl} are the elastic parameters, ρ is the density, and ω is the angular frequency. Throughout this study we will follow the convention that a lowercase subscript takes the values of 1, 2, and 3 while an uppercase subscript takes only the values of 1 and 2. In the zero-order ray method, we seek an approximate solution to (1) in the form of $u_k(x_i,\omega)=U_k(x_i)e^{i\omega\tau(x_i)}$, where $U_k(x_i)$ and $\tau(x_i)$ are, respectively, the amplitude and traveltime along the ray. Substituting this ray solution into (1) and letting $\omega\to\infty$ yields the Christoffel equation:

$$\left(\Gamma_{jk}-\delta_{jk}\right)U_k=0, \qquad (2)$$

where the Christoffel matrix $\Gamma_{jk}=a_{ijkl}p_i p_l$ with the density normalized elastic parameters $a_{ijkl}=c_{ijkl}/\rho$, and the slowness vector $p_i=\partial\tau/\partial x_i$. Equation (2) is an eigenvalue problem and its eigenvalues take the form:

$$G(x_i,p_i)=1 \qquad (3)$$

which solves the eigenvalue equation

$$\left(\Gamma_{jk} - G\delta_{jk}\right)g_k = 0, \tag{4}$$

where g_k is the normalized eigenvector and often referred to as the polarization vector. Multiplying (4) with g_j and taking into account that $g_k g_k = 1$, we obtain

$$G = \Gamma_{jk} g_j g_k = a_{ijkl} p_i p_l g_j g_k. \tag{5}$$

Since $p = \nabla\tau$, equation (3) is a nonlinear first-order partial differential equation for the phase function or eikonal $\tau(x_i)$ which describes propagation of a P- or S-wave wavefront. This eikonal equation can be solved using the Hamiltonian (Cerveny, 1972)

$$H(x_{i,} p_i) = \frac{1}{2}\left(G(x_i, p_i) - 1\right), \tag{6}$$

yielding the kinematic raytracing system for P or S waves:

$$\frac{dx_i}{d\tau} = \frac{\partial H}{\partial p_i} = \frac{1}{2}\frac{\partial G}{\partial p_i} = a_{ijkl} p_l g_j g_k \tag{7a}$$

$$\frac{dp_i}{d\tau} = -\frac{\partial H}{\partial x_i} = -\frac{1}{2}\frac{\partial G}{\partial x_i} = -\frac{1}{2}\frac{\partial a_{mjkl}}{\partial x_i} p_m p_l g_j g_k. \tag{7b}$$

Equations in (7) are complicated and inefficient for computation as they require solving the eigenvalue problem (4) at each ray step. Efforts have been made to simplify equations (7). Cerveny (1989), for example, has found that (7) can be greatly simplified by using factorized anisotropic media, but such a medium requires that the relative spatial variations be identical for all elastic parameters, limiting the application of this approach for inhomogeneous media. Equations (7) also require the medium to be specified in terms of elastic parameters. This may result in ambiguity in specifying parameters, for example, for the widely used weak TI media. For P wave imaging in such a medium, usually only vertical P-wave velocity α_0 and Thomsen parameters δ and ε are estimated from data. Determination of the elastic parameters from parameters α_0, ε and δ, on the other hand, requires explicit knowledge of S-wave vertical velocity β_0.

To overcome these difficulties, we reformulate the ray equations in (7) in terms of phase velocity in the same way as the ray equations in isotropic media are formulated (e.g., Zhu and Chun, 1994). The only difference is that the phase velocity in isotropic media is equal to the group velocity and is simply referred to as velocity. To accomplish this, we first note that Geoltrain (1989) has shown that the group velocity for energy propagation along the x_i direction is given by $V_i = a_{ijkl} p_l g_j g_k$. Thus equation (7a) can be rewritten as $dx_i / d\tau = V_i$, showing that wave energy propagates along ray **x** = (x_1, x_2, x_3). To simplify equation (7b), we take into account the fact that the eigenvalue G in (5) and its derivative $\partial G/\partial x_i$ are both homogeneous functions of the second degree in p_i. This leads to

$$v^2 = G(x_i, n_i) \quad \text{and} \quad \frac{\partial G(x_i, p_i)}{\partial x_i} = \frac{1}{v^2}\frac{\partial G(x_i, n_i)}{\partial x_i} = \frac{2}{v}\frac{\partial v}{\partial x_i} = 2\partial \ln v / \partial x_i, \tag{8}$$

where n_i is the unit vector along the slowness vector p_i and $v = v(x_i, n_i)$ is the phase velocity. A substitution of the second equation in (8) into (7b) enables us to rewrite the kinematic raytracing system (7) in terms of phase velocity:

$$dx_i / d\tau = V_i \tag{9a}$$

$$dp_i / d\tau = -\partial \ln v / \partial x_i \tag{9b}$$

where group velocity V_i is calculated from phase velocity with the formulae given, for example, by Tsvankin (2001).

The ray tracing system in (9) is simpler than that in (7) and takes the same form as its counterpart in isotropic media except that the right-hand side of (9a) is now given by V_i instead of v_i as in isotropic media, showing that wave energy in an anisotropic medium no longer propagates along the direction of the slowness vector due to the angular dispersion induced by the anisotropy. System (9) is also computationally more efficient than (7), especially for TI and orthorhombic media as the derivatives in (9) can be evaluated quickly with the expressions for phase-velocity given by Thomsen (1986) and Tsvankin (2001). Solution of eigenvalue problem (4) at each ray step is no longer needed. As the medium for raytracing is now specified with phase velocity, the ambiguity problem in specifying elastic parameters for weak TI media is eliminated.

Dynamic raytracing system

Dynamic raytracing was originally proposed for ray-amplitude calculation (Cerveny, 1972; Kendall and Thomson, 1989). Since then, it has found many other applications in seismic modeling and imaging (e.g., Alkhalifah, 1995; Cerveny, 2001). For example, it has been used to construct Gaussian-beam solutions to equation (1). Dynamic raytracing equations in anisotropic media are commonly expressed in Cartesian coordinates (e.g., Cerveny 1972). This leads to a system of six linear first-order ordinary differential equations. For many applications, it is convenient to use ray-centered coordinates. The dynamic raytracing system also takes the simplest form in such coordinates, reducing the number of differential equations in the system from six to four. Here we will first formulate dynamic raytracing equations in terms of phase velocity in Cartesian coordinates, and then transform them to a ray-centered coordinate system.

Consider the ray coordinates $(\gamma_1, \gamma_2, \tau)$ where γ_1 and γ_2 are the ray parameters that specify a ray and τ is the traveltime along the ray. The dynamic raytracing system in Cartesian coordinates can then be obtained by differentiating the ray equations in (9) with respect to the ray parameters. This gives

$$dQ_i / d\tau = A_{ij} Q_j + B_{ij} P_j, \qquad dP_i / d\tau = -C_{ij} Q_j - D_{ij} P_j, \tag{10}$$

where $Q_i = \partial x_i / \partial \gamma$, $P_i = \partial p_i / \partial \gamma$, and γ represents γ_1 or γ_2. The coefficients A_{ij}, B_{ij}, C_{ij} and D_{ij} are given by

$$A_{ij} = \partial V_i / \partial x_j, \qquad B_{ij} = \partial V_i / \partial p_j \tag{11a}$$

$$C_{ij} = \partial^2 \ln v / \partial x_i \, \partial x_j, \qquad D_{ij} = \partial^2 \ln v / \partial x_i \, \partial p_j. \tag{11b}$$

Ray-centered coordinates in anisotropic media (y_1, y_2, τ) have been used by Hanyga (1986) and Cerveny (2001) to derive dynamic raytracing equations. The coordinates are defined along a reference ray, often referred to as the central ray. Coordinate τ is the traveltime along the central ray and coordinates y_1 and y_2 lie in the plane tangential to the wavefront at ray point τ. Their corresponding basis vectors are $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ with

$$\mathbf{e}_3 = (V_1, V_2, V_3). \tag{12a}$$

The ray-centered coordinates in anisotropic media are nonorthogonal as the ray is no longer perpendicular to the wavefront as it is in an isotropic medium. Different approaches have been used by Hanyga (1986) and Cerveny (2001) for defining the plane basis vectors $\mathbf{e}_I$. As an alternative, we define them along the central ray by the differential equations

$$d\mathbf{e}_I / d\tau = v(\mathbf{e}_I \bullet \nabla v)\mathbf{p}. \tag{12b}$$

Equations in (12b) are the same as those used by Popov and Psencik (1978) for defining the plane basis vectors of the ray-centered coordinates in isotropic media. Compared to the previous choices by Hanyga (1986) and Cerveny (2001), the ray-centered coordinate system defined by (12) has the advantage in that it reduces to the ray-centered coordinate system described by Popov and Psencik (1978) in an isotropic medium. Moreover, its plane basis vectors $(\mathbf{e}_1, \mathbf{e}_2)$ are orthonormal in the tangential plane and can therefore be determined by integrating just one of them along the central ray.

Using the definitions of the basis vectors given in (12), the transformation matrix from the ray-centered coordinates to the Cartesian coordinates T_{ij} can be written as

$$T_{iK} = e_{Ki}, \qquad T_{i3} = e_{3i} = V_i, \tag{13}$$

where e_{Ki} are the Cartesian components of basis vectors $\mathbf{e}_K$ with K = 1, 2. We also denote the slowness vector in this ray-centered coordinate system by $q_I = \partial \tau / \partial y_I$. Using the transformation matrix (13) and its inverse, and following the approaches used by Hanyga (1986) and Cerveny (2001) for deriving their the dynamic ray equations in the ray-centered coordinates, we obtain from (10):

$$d\overline{Q}_I / d\tau = A_{IJ}\overline{Q}_J + B_{IJ}\overline{P}_J, \qquad d\overline{P}_I / d\tau = -C_{IJ}\overline{Q}_J - D_{IJ}\overline{P}_J, \tag{14}$$

where $\overline{Q}_I = \partial y_I / \partial\gamma$, $\overline{P}_I = \partial q_I / \partial\gamma$. The coefficients in (14) are given by

$$A_{IJ} = \partial(\partial \ln v / \partial y_J) / \partial q_I - v(\partial v / \partial y_J)\overline{p}_I, \qquad B_{IJ} = \partial \overline{V}_J / \partial q_I \tag{15a}$$

$$C_{IJ} = v^{-1}\partial^2 v / \partial y_I \partial y_J, \qquad D_{IJ} = \partial(\partial \ln v / \partial y_I) / \partial q_J - v(\partial v / \partial y_I)\overline{p}_J, \tag{15b}$$

where $\overline{V}_I$ and $\overline{p}_I$ are, respectively, the components of the group velocity vector **V** and slowness vector **p** in the ray-centered coordinate system. The dynamic raytracing system in (14) consists of four linear first-order ordinary differential equations and is similar to that in isotropic media except that it contains two extra terms with respective coefficients A_{IJ} and D_{IJ} due to the effects of anisotropy. For an isotropic medium, these coefficients vanish and (14) reduces to the well-known dynamic raytracing system in the isotropic ray-centered coordinates (Popov and Psencik, 1978).

Dynamic raytracing systems (10) and (14) are much simpler and computationally more efficient than those formulated in terms of elastic parameters (e.g., Cerveny, 1972; Hanyga, 1986; Kendall and Thomson, 1989) since the elastic parameter based formulations involve differentiation of the complicated functions on the right-hand sides of equations (7) with respect to ray parameters. Evaluations of the right-hand sides of dynamic raytracing systems in (10) and (14), on the other hand, are relatively simple, especially for TI and orthorhombic anisotropic media where simple analytic expressions for phase velocity have been derived by Thomsen (1986) and Tsvankin (2001).

Conclusion

We have developed new systems for kinematic and dynamic ray tracing in inhomogeneous, anisotropic media. Formulated in terms of phase velocity, these systems are simpler and computationally more efficient than previous elastic parameter based raytracing systems (e.g., Cerveny, 1972; Hanyga, 1986; Kendall and Thomson, 1989; Cerveny 2001), especially for the dynamic raytracing system. The previous dynamic raytracing systems involve differentiation of the complicated functions on the right-hand sides of equations (7) with respect to ray parameters while systems (10) and (14) require only simple evaluation of the derivatives of phase and group velocities. The new kinematic and dynamic systems also have the advantage in that the medium used for raytracing is now specified with phase velocity, eliminating the need to calculate elastic parameters from Thomsen parameters and hence the possible ambiguity in specifying elastic parameters for P-wave imaging in weak TI media.

The efficiency of our dynamic raytracing is further enhanced by the introduction of the nonorthogonal ray-centered coordinate system (12). Determination of the plane basis vectors of this coordinate system requires only a single integration along the central ray, and the number of dynamic ray equations in the coordinate system reduces from six to four. This dynamic raytracing system, coupled with kinematic ray equations in (9), thus provides an efficient and useful tool for seismic modeling and imaging in anisotropic media. These systems are especially useful for the TI and orthorhombic media where the right-hand sides of equations in (9) and (14) can be evaluated relatively inexpensively with the analytic expression for phase velocity given by Thomsen (1986) and Tsvankin (2001).

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