Anisotropy kinematics ray models: are they consistent with Fermat's principle and Snell's reflection/ refraction laws?

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Summary

anisotropy kinematics un-puzzled: view p.4

Wave disturbance progression is key for probing the unseen in our earth. More than a century back wave theory and elasticity physics researchers discerned that ordered fine structure medium detail, we call it now anisotropy, impacts on disturbance propagation. The theme was researched already in the 18-hundreds, eg. Hamilton, McCullagh, Green . Some manifestations have still not been fully explained and credibly modeled. I clarify here one of the long-puzzling issues: in uniform anisotropic media, how do the direction- and velocity-distinct (*so-called*)-*RAY*s [viz. *group*- or *ray*-velocity loci] progress the wavefronts in *NOT-tangent-plane-orthogonal* directions? Through in energy-flux-channels interior-linked *Fermat-Snell*-premises-consistent *segmentals* ! Re-ordering as *regime rays* clarifies *FS-rays* detail and front progression impact.

Theme issues and insights

In an 1987 SEG 'Leading Edge' historical discourse Enders Robinson and Dean Clark remind us (re Pierre de Fermat, 1601-1665) that "… what is now called *Fermat's Principle* or the principle of least time …. is absolutely indispensable to our current concept of seismic wave propagation". Their recap includes the (prior to Snell) ray-theory framed *law of reflection* pondered already in antiquity, and the *Snell's law of refraction* re direction changes at smooth interfaces between distinct homogeneous medium segments [1621(?), Willebrord Snell (1580-1626)]. Those formulations have been generalized for *ray* loci in media broadly, for smoothly heterogeneous media and smooth interfaces. But do they encompass anisotropic media?

As elastic media theory and disturbance progression became progressively better understood and more rigorously analytically framed, the strict sense *FS-rays* (*Fermat's* and *Snell's* premises combined) were challenged to explain wave kinematics in anisotropic media [viz. restricted here to uniform anisotropy at heterogeneity scale greatly less than wavelengths in the disturbance signals]. Re historical background, Helbig (1978, 1994) points to early publications by Hamilton (1837) and McCullagh (1837), Rudzki 1911, and many others. Other valued publications include (Postma 1955, Gassmann 1964, Vlaar 1968, Auld 1973, Berryman 1979, Thomsen1986, 2002, Winterstein 1990, Helbig 1994, McBeth and Lynn 2000, Cerveny 2001, Slawinski 2002, … …).

Can we model *ray* and wavefront kinematics in anisotropic media within the frame of just the trusted *FS-ray* premises, and will so-framed models encompass and shed light on so-called *long-wavelength regime* wavefront progression detail? I suggest they can be so framed, but I contend also that as yet they have not been correctly so framed (Vetter1993, 1999, 2004, 2007, 2009), contrary to current thinking broadly. I highlight here principal issues and insights:

(1) the plane wave model is an *ansatz* (starting premise), [c.f. Sheriff 1984 encyclopedic dictionary, Aki/ Richards 1980, …]: '*a wavefront with no curvature, as in a homogeneous medium might stem from a remote point source* $\boldsymbol{r}_O$ *at coordinate frame origin*'. Its principal feature is that along front-normal $\boldsymbol{n} = [n_x\ n_y\ n_z]$ oriented *rays* $\boldsymbol{r}_{ON} = \boldsymbol{r}_N - \boldsymbol{r}_O$ have progression speeds V_N, whereas then other *rays* $\boldsymbol{r}_{OE} = \boldsymbol{r}_E - \boldsymbol{r}_O$ to points on the planewave front per se have APPARENT progression speeds V_E that are greater than those of front-normal V_N.

The analytical pathtimes *ansatz* $t_{ON} = t_{OE} = (n_x x + n_y y + n_z z)/ V_N$ is invoked as an anisotropy kinematics premise whereby, for particular $t_{ON} = t_{OE}$, the $\boldsymbol{r}_{ON}$ and $\boldsymbol{r}_{OE}$ are paired in the plane-wave sense; but now with concretely manifesting $\{ \boldsymbol{r}_{OE}, V_E \}$ and linked not tangibly manifesting $\{ \boldsymbol{r}_{ON}, V_N \}$. For firmed $t_{ON} = t_{OE}$ instant, all such paired points firm the $r_{E\text{-}SURFACE}$ wavefront, and the associated *front-normal* representation surface $r_{N\text{-}SURFACE}$. $\boldsymbol{r}_{OE}$ project orthogonally onto their paired $\boldsymbol{r}_{ON}$ (on representation front-normal surface). This is credible adaptation of *plane-wave* rooted detail in uniform anisotropy for paired { $\boldsymbol{r}_{ON} = \boldsymbol{r}_N - \boldsymbol{r}_O$, V_N } and { $\boldsymbol{r}_{OE} = \boldsymbol{r}_E - \boldsymbol{r}_O$, $V_E > V_N$ } .

(2) ***rays*** [*RAY*s, above $\boldsymbol{r}_{OE}$] has become anisotropy-context designation for 'as line loci idealized energy flux channels' (so-called *group*- or -*RAY*-velocity loci). But we ponder still why the long-honoured premise that "***rays*** [*FS-rays* implicit] progress orthogonal to their wavefront tangent planes" does not hold for *anisotropic media* (cf. Cerveny 2001, p103, *ray* premises)! Rather, the fronts here progress in *NOT-tangent-plane-orthogonal* directions. Could that be because (*so-called*)-*RAY*s might be *NOT*-*FS-rays*, which calls for further pondering re kinematics detail?

And what about front triplications (so-called cusping); are they credibly explained and modeled? [c.f.Vetter 1993, qSH-mode simuln-example]. How can it all come about through {$\boldsymbol{r}_{OE}$, V_E } loci?

Adding to this puzzling is that reflection/ transmission/ refraction- re interfaces between two anisotropic medium segments must supposedly involve not directions/ progression speeds of the (*so-called*)-*RAY*s, but rather those of thereto linked front-normals. How would those front-normal{ $\boldsymbol{n}$, V_N } even get to the proper transition points! All the above are quite troubling issues.

Can we fathom ray theory framed kinematics for anisotropic media as just categorically invalid? Is anisotropy so exceptional in the realm of physics that the '*principle of least action*', (thereby *Fermat's principle* and *Snell's laws* also encompassed), would categorically not apply for wave disturbance progression framed by ray theory models? Or rather, **might current thinking be just incorrect and models incomplete ?**

(3) 'incomplete models': for uniform anisotropic medium context we focus on progression detail along the straight line $\boldsymbol{r}_{OE}$ loci, viz. the as-line-idealized energy flux channels. Speeds along $\boldsymbol{r}_{OE}$ become tangible through {$\boldsymbol{r}_{OE}$, t_{OE}} pairing, with then {$V_E = v_{APPARENT}$ =| $\boldsymbol{r}_{OE}$ |/ t_{OE} } as so-called *group*- or *RAY*-velocities. But what firms the loci-specific ($\boldsymbol{r}_{OE}$, t_{OE} , v_{APRNT})? It would be the structured heterogeneity encountered along the direction-specific energy flux channels in the uniform anisotropy, i.e. vicinity around the $\boldsymbol{r}_{OE}$ loci! [c.f. Thomsen 2002, p.1-5: " … … **ordered heterogeneity on the small scale** … …**appears as anisotropy at the large scale**"].

When medium is perturbed through long-wavelengths probing, heterogeneity specific to causative uniform anisotropy in not-smooth-walled $\boldsymbol{r}_{OE}$-channels impacts on channel-interior disturbance progression. Suppose that *FS-ray theory* was here relevant, say at scale of *ray*-segmentals with lengths perhaps on order of medium detail repetition, and with appropriate $\boldsymbol{r}_{OE}$ -channel area to give the segmentals adequate direction flexibility. Is there then hope and scope for retrieving transparent information re underlying *FS-rays*? How exactly do elasticity theory rooted {$\boldsymbol{n} = [n_x\ n_y\ n_z]$, $\boldsymbol{r}_{ON} = \boldsymbol{r}_N - \boldsymbol{r}_O$, V_N } impact on energy-flux { $\boldsymbol{r}_{OE}$, V_E }-loci orientations and progression kinematics through here uniform anisotropic medium segments?

All this becomes transparent through what I call ***regime rays***. Those are aggregation of same-oriented *segmentals* to single *segments*. For uniform anisotropy of likely all categories, for pure qP, qSV, qSH propagation modes, but just direct-forward progression, the *segmentals* re-order to just three linked representation *segments*. These manifest *segmentals* directions and speeds, but not the lengths. We discern $\boldsymbol{r}_{OE}$ -wise *FS-ray*-distribution velocities {v_{APRNT} , $v_{TIME\text{-}AVE}$, v_{RMS}, $v_{PATH\text{-}MEAN}$ }, also then quantified $\boldsymbol{r}_{OE}$ -heterogeneity. Such $\boldsymbol{r}_{OE}$ -loci detail, possibly as average-compacted over front-portions or areal sensor spreads, will be invaluable re anisotropy impact.

Example visualized

Page-4 (c.f. Vetter 2007, 2009) adds concrete visualization. It shows qSH-mode of Schoenberg-Helbig's (1997) "standard model". Think cross-diagonal line between [1.5 0 0] and [0 1.5 1.5], with six at equi-angle spaced $\boldsymbol{n} = [n_x\ n_y\ n_z]$; the $\boldsymbol{n}$-directions are spread out for broad scan through the medium. $\boldsymbol{n}$-directions plus elastic constants/ density firm front-normal speeds V_N . Further, $t_{ON} = t_{OE} = 1$ sec. [41 dots, 40-equi-time intervals] firm oriented $\boldsymbol{r}_{ON}$ (blue) and $\boldsymbol{r}_{OE}$ (red).

Except for #1*ray*, 1st-regime is dotted part of $\boldsymbol{r}_{ON}$. The remainder blue lines refract by *Fermat*- and *Snell 's* dictates, thus segments 2 and 3 of *regime representation rays*. The 4D-view shows slow/fast of oriented segments [dots dense/sparse]. Ponder #3 to #4 *regime*-transition (a switch 'in-micro-layers dominant directional characteristics?!), almost cusping-like if ***n***-directions were dense. All such suggests potential for discerning certain fine medium detail through *regime rays*!

FS-rays progression visualized: Consider any specific *regime ray* with its complete associated detail, but all that together with as twice- mirror-image-flipped complement. Thereby straight-line $\boldsymbol{r}_{OE} = \boldsymbol{r}_E - \boldsymbol{r}_O$ with its time-ticks is doubled. Then through replication of the regime ray segments, what was at coordinate frame origin has again point-coincidence with $\boldsymbol{r}_{OE}$ -progressed. There are now five segments (mid-segments at double lengths); so manifest the *FS-ray*-progression patterns. ***Regime rays* display/ encapsulate analytically/ what *FS-rays* do in the small** !

Conclusions

Regime rays for anisotropy, re-firm that Fermat's principle and Snell's laws (as Enders R. and Dean C. framed it back in 1987, [elaborated]) are indeed "… absolutely indispensable to our **now**-current **anisotropy-inclusive** concept of seismic wave propagation". Long back (perhaps Rudzki 1913 ?), {$\boldsymbol{r}_{OE}$-,V_E}-loci/ -channels in anisotropic media were deemed(!!) /called ***rays***, an improper designation! Call them tentatively *AnisoFS*- or *AFS*-channels?? The deliberations here should contribute to ***INFORMED-convinced*** response to the *Abstract*-title(**?**), "**YES they are!**".

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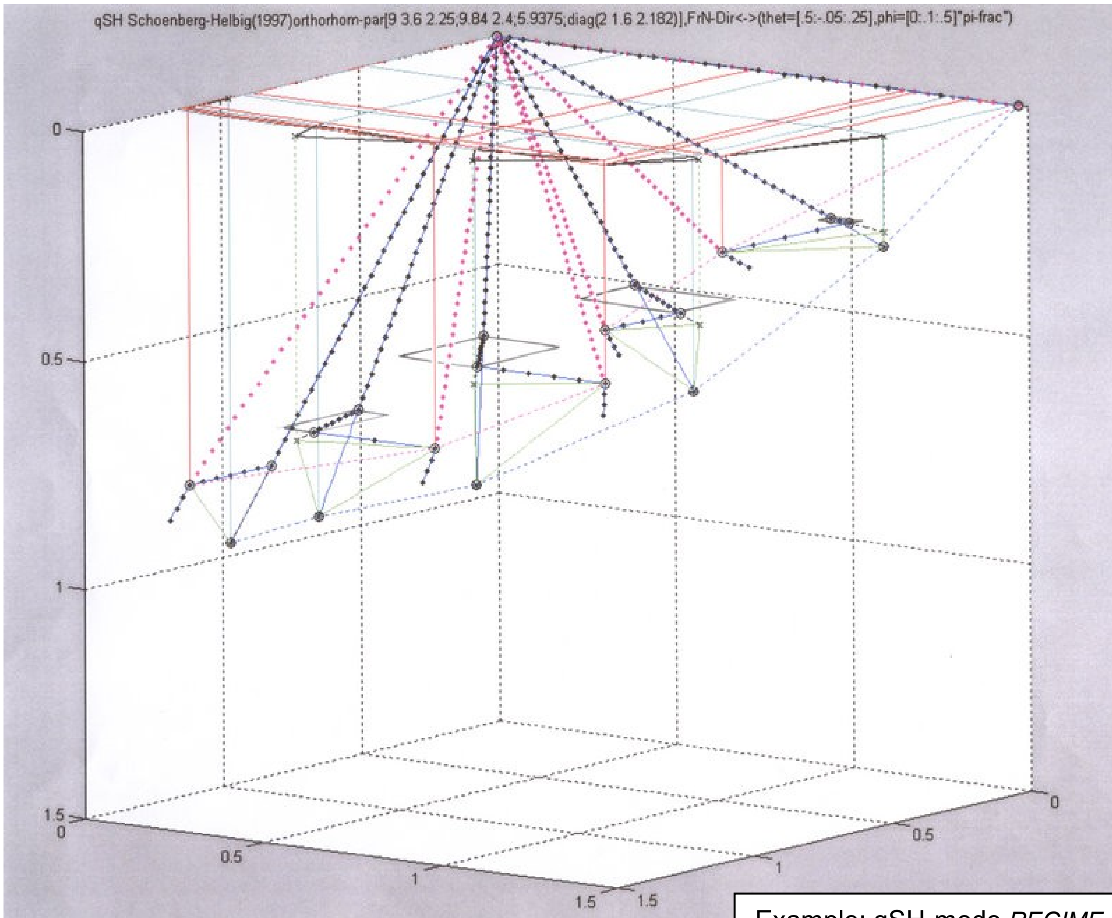

r_{OE} energy flux, 41 red equi-time dots is one second; they are '*NOT-RAY*' channels for *FS-ray* progression

r_{ON} so-called front-normal- or *phase- or group-velocity*-loci; blue-dotted r_{ON} -portion are N-direction/time for 1st *regime-ray* segments
black-framed rectangles: part of x-y surface at z-depths; dots show direction/time for 2nd *regime-ray* segments; NOTE N-dirctn refracts to in-rectangls 2nd segments; then rectangls edge extended as *regime-ray* 3rd sgmnts NOTE 2nd-dirctn refracts to beyond-rectngl-edge line; dense dots is slow & sparse dots is faster to very fast eg.#2 & #5 *regm-ray* (view Data#2 #4 segmnt-*velos*)
r_{ON} – r_{OE} associatn: r_{ON} refracts parts to 2nd/ 3rd *regimes*! explains long-puzzlng *phase*-velo/ *group*-// *ray*-velo link

green triangls are r_{ON} -orthagonal tangent plane portions firmed by { r_{ON} , r_{OE} and regm2-segment-extended}

'***regimes***' here for six *rays*: {x NZy NZy NZx NZx Xy} with **N** for in3D; **Z** & **X** for in-planes; **x** & **y** for on-lines

Regime rays model and 4D(space-time)-display the *FS-ray* kinematics in uniform anisotropic media

Example: qSH-mode *REGIME RAYS*
Schoenberg-Helbig orthorhrombic model
Data for # 2 & # 4 ***FS-rays***

2 { n_E , n_N }directns *time* *length* *velo*
E[. 8236 . 5520 . 1302] 1. 000 1. 6154 1. 6154
N[.9393 .3052 .1564] .8647 1.3445 1. 5548
Z[.9511 . 3090 0] .0451 . 0711 1. 5742
y[0 1.0000 0] . 0902 . 4595 5. 0942
vTA = 1.8749 vRMS = 2.1314 vPM = 2. 4229
len*FS-ray* =1.8749 len*NOT-ray*-OE = 1. 6154
P[-. 2814 . 9576 . 0623] Polarization________

4 { n_E , n_N }directns *time* *length* *velo*
E[. 6945 . 6568 .2938] 1. 000 1. 6571 1. 6571
N[.5237 . 7208 .4540] .6668 1.0724 1. 6083
Z[.5878 . 8090 0] .2160 .3898 1. 8050
x[1.0000 0 0] .1172 .3601 3. 0709
vTA = 1.8222 vRMS =1.8799 vPM = 1.9393
len*FS-ray* =1. 8222 len*NOT-ray*-OE = 1.6083
P[-. 8412 . 5386 -. 0479] Polarization________