Split-step Two-way phase-shift time stepping for wavefield propagation

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Abstract

We consider a time-marching algorithm to numerically solve the acoustic wave equation. Unlike finite-difference solvers, our method is not dispersive, and is based an analytic solution to the constant velocity wave equation. The analytic solution is too numerically complex to calculate and so is approximated with a Taylor-series expansion. The computational properties of the approximate solver are similar to higher-order in time pseudospectral methods.

Introduction

Reverse-time migration (RTM) and forward modelling by differencing the two-way acoustic wave equation (Baysal et al., 1983; McMechan, 1983) are computationally expensive. However with an accurate velocity model they are very effective methods for migration and modeling. We consider a number of alternative methods to solve the acoustic wave equation in the wavenumber domain.

Pseudospectral Methods

Pseudospectral methods are numerically efficient methods to solve the full two-way acoustic wave equation. They compute the spacial Laplacian exactly by using a Fourier transform and as a result they allow larger spactial sampling rates. The symbols $\mathcal{F}_{\vec{x}\to\vec{k}}$, and $\mathcal{F}^{-1}_{\vec{k}\to\vec{x}}$ are used to denote the forward and inverse Fourier transforms with respect the spactial and wavenumber variables, respectively. The acoustic constant-density variable-velocity wave equation is

$$\begin{cases} \frac{\partial^2 U}{\partial t^2} = v^2\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial z^2}\right) \\ U(0,\vec{x}) = f(\vec{x}), U(-\delta t,\vec{x}) = g(\vec{x}) \end{cases}, \tag{1}$$

where $U(t,\vec{x})$ is the amplitude of the wave at the point $(t,\vec{x})$, x is the lateral coordinate, z is the depth coordinate, t is the time coordinate, $\partial^2 U/\partial t^2$ is, for example, the second-order partial derivative of the wavefield with respect to the time coordinate, and v, is the speed of propagation. Assume $\vec{x} \in \mathbb{R}^2$ and $t \in \mathbb{R}$. The pseudospectral time-marching algorithm is

$$U^{n+1} = 2(U)^n - U^{n-1} - \delta t^2 v^2 \mathcal{F}^{-1}_{\vec{k}\to\vec{x}}[(2\pi|\vec{k}|)^2 \mathcal{F}_{\vec{x}\to\vec{k}}[U]], \tag{2}$$

where the superscripts n refers to the approximation at timestep n. The Fourier transform is used to calculate the Laplacian and a second-order finite-difference operator is used to

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calculate the time derivative. A higher-order finite-difference approximation for the second-time derivative of U is unconditionally unstable (Cohen, 2001). Alternatively, the modified equation approach (Cohen, 2001) can be used to calculate the time derivative more precisely. The Taylor series expansion of the second-order time derivative is

$$\left(\frac{\partial^2 U}{\partial t^2}\right)^n = \frac{U^{n+1} - 2U^n + U^{n-1}}{\delta t^2} - \frac{\delta t^2}{12}\left(\frac{\partial^4 U}{\partial t^4}\right)^n + O(\delta t^6). \tag{3}$$

Substituting equation (3) into the scalar wave equation gives the fourth-order time approximation. If this procedure is iterated, then formally (Etgen, 1989; Dablain, 1986),

$$U^{n+1} = -U^{n-1} + 2\sum_{k=0}^{\infty} \frac{(\delta t v)^{2k}}{(2k)!}\left(\Delta^k U\right)^n \tag{4}$$

where ΔU refers to the 2-dimensional Laplacian of the function U and $\Delta^2 U$ is the biharmonic or the Laplacian applied twice to U. Taking the Fourier transform of both sides of equation (4) with respect to the spatial coordinates gives

$$\begin{aligned} \widehat{U}^{n+1} &= -\widehat{U}^{n-1} + 2\sum_{k=0}^{\infty} \frac{(\delta t v)^{2k}}{(2k)!}\left((-2\pi|\vec{k}|)^{2k}\widehat{U}\right)^n \\ &= -\widehat{U}^{n-1} + 2\cos\left(2\pi v|k|\delta t\right)\widehat{U}^n. \end{aligned} \tag{5}$$

Split-Step Time Stepping

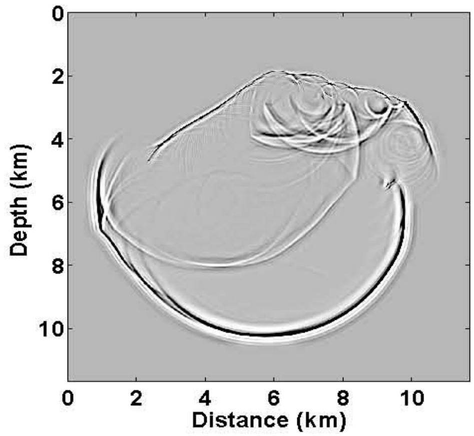
Equation (5) is a dispersion free method of solving the acoustic wave equation and for constant velocity is an exact solution. However it is expensive because a fast Fourier transform cannot be used to calculated the inverse Fourier-like transform. A Taylor series can be used to approximate the variable velocity cosine operator about the reference velocity v_0. If a Taylor series expansion is used about $v_0 = 0$ then the higher-order in time pseudospectral methods is derived. The power series expansion about the velocity v_0 with the variation $\delta v = v(x) - v_0$ for the function $\cos(2\pi v(\vec{x})|\vec{k}|\Delta t)$ is

$$\begin{aligned} \cos(2\pi v(\vec{x})|\vec{k}|\Delta t) &= \cos(2\pi v_{ref}|\vec{k}|\delta t) \\ &- \sin(2\pi v_{ref}|\vec{k}|\delta t)\delta v(\vec{x})2\pi|\vec{k}|\Delta t \\ &- \frac{1}{2}\cos(2\pi v_{ref}|\vec{k}|\delta t)\left[\delta v(\vec{x})2\pi|\vec{k}|\Delta t\right]^2 + H.O.T. \end{aligned} \tag{6}$$

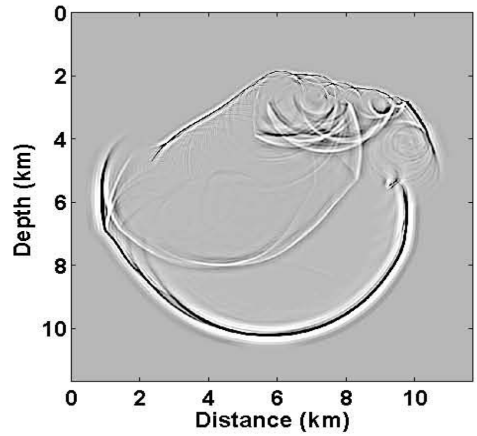
where H.O.T. denotes higher order terms. Substituting the Taylor series expansion (6) into equation (5) gives the second-order splitstep correction. For large velocity variations δv, the split-step correction can become inaccurate and unstable. The timestep must satisfy $\delta t < \delta x/\sqrt{2}V_{max}$ due to aliasing considerations.

Numerical Examples

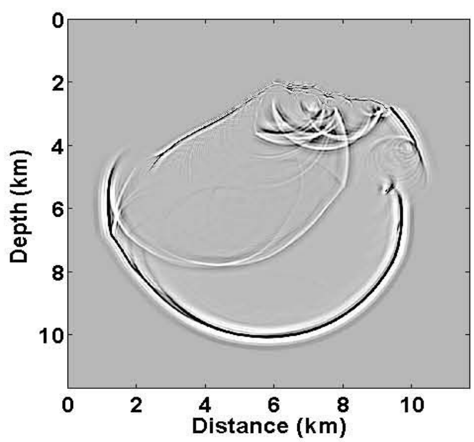
We compare pseudospectral methods to split-step PSTS methods by looking at some snapshots of a forward propagated wavelet, injected at the center of the model, through a portion of the BP data set in Figure 1(e). The BP data set contains a rigorous saltdome embedded



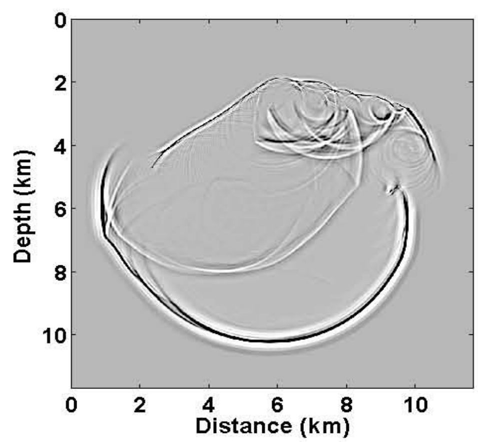
(a)



(b)



(c)



(d)

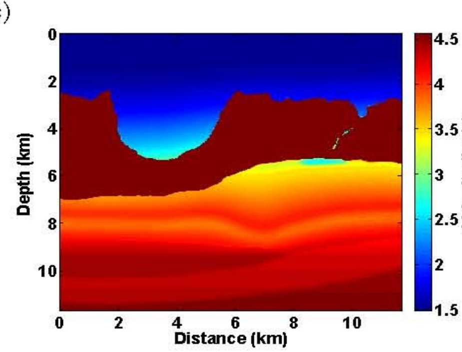


Figure 1: (a) Second-order pseudo-spectral method. (b) Forth-order pseudo-spectral method. (c) First order split-step PSTS with one window. (d) Second order split-step PSTS with one window. (e) A section of the BP data set showing the rigours salt dome.

algorithm	Relative time	time step (ms)	grid spacing	number of FT
pseudo 2nd order	0.8	1.2	12.5	3
pseudo 4th order	1.0	1.5	12.5	3
splitstep 1st order	0.9	1.5	12.5	3
splitstep 2nd order	1.3	1.5	12.5	4

Table 1: Relative computation time and timestep size used to make Figure 1

in a background sediment whose velocity smoothly increases with depth. Figure 1(a) and (b) is the snapshot using second-order pseudospectral method derived in equation (2). The lower-order method is computationally efficient but contains unacceptable dispersion while there is no observable dispersion in the higher order implementation. Figure 1(c) is the first order split-step correction. The snapshot is not dispersive but there are large kinematic errors due to the low order of the approximation. Figure 1(d) is the second order split-step correction. The kinematics are much better than in Figure 1(c) but there is more dispersion than the fourth-order pseudospectral method.

Conclusion

We presented a new method to numerically solve the acoustic wave equation. It is similar to higher-order in time pseudospectral methods based on the modified equation approach or the Lax-Wendroff method. This approximation scheme can be used for acoustic modeling or reverse time migration. If there is a small variation in the velocity off a reference velocity then the method is superior to higher-order pseudospectral methods.

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