Tensor unfolding principles and applications to rank reduction reconstruction and denoising

Nadia Kreimer, University of Alberta, Edmonton, Canada
kreimer@ualberta.ca
and
Mauricio D. Sacchi, University of Alberta, Edmonton, Canada

GeoConvention 2012:Vision

Summary

Seismic data can be represented by a N-dimensional structure or tensor that can be unfolded in N matrices. These N unfolded matrices (also called unfoldings) are low rank when the data are composed of a linear superposition of linear events. Noise and missing observations increase the rank of the unfolded matrices and, therefore, iterative rank reduction of these N unfolded matrices permits to recover missing traces and to enhance the signal-to-noise ratio of the seismic data.

Introduction

Reconstruction of pre-stack seismic data has received attention in recent years because seismic acquisition rarely leads to fully sampled wave-fields. Multi-dimensional reconstruction, commonly named *5D interpolation*, offers a remedy to this problem.

Rank reduction methods that exploit the low dimensionality of seismic data were proposed by Freire and Ulrych (1988), Trickett (2008), Trickett et al. (2010) and Oropeza and Sacchi (2011). Methods that directly operate on the seismic data matrix, often referred as eigen-image filtering, have been proposed to denoise data in $t-x$ (Freire and Ulrych, 1988) and $f-x-y$ (Trickett, 2008). Recent generation of methods operate on multi-level Hankel or Toeplitz matrices (Trickett et al., 2010; Gao et al., 2011). At the core of these methods is rank reduction implemented via the SVD or Lanczos decomposition. The basis idea is that properly sampled seismic data (or its Hankel matrix) in the absence of noise are a low rank structure. Noise and unrecorded data will increase the rank of the data matrix or its Hankel matrix. Therefore, denoising and reconstruction is easily implemented via iterative rank-reduction (Oropeza and Sacchi, 2011).

The aforementioned methods operate on matrices or on multi-dimensional structures transformed into multi-level Hankel matrices. Recently, rank reduction methods that directly operate on tensors have been proposed to solve the multi-dimensional denoising and reconstruction problem (Kreimer and Sacchi, 2011). Tensor decompositions offer an alternative way of rank reduction. Particularly, the Higher-Order Singular Value Decomposition (HOSVD) (De Lathauwer et al., 2000) leads to a reconstruction algorithm where there is no need of embedding the multi-dimensional seismic data in multi-level Hankel matrices. In this paper, we investigate yet a new method that operates directly on the seismic tensor. Sequential rank reduction of unfolded matrices leads to an iterative algorithm with properties very similar to that of the HOSVD.

Theory

A simple algorithm for rank reduction of seismic tensors

Consider a seismic volume with 3 spatial coordinates $D(t,x,y,z)$. The algorithm can be easily generalized to data that depends on four dimensions. The DFT can be used to transform the volume to the $f-x-y-z$ domain. For each temporal frequency f, this volume can be arranged as a 3rd order tensor that depends on x,y,z. A 3rd order tensor can be unfolded in three matrices. Each unfolding is a rearrangement of the slices of the tensor in different directions (or modes) into a matrix.

Assuming that each one of the unfolded matrices obtained form the tensor $\mathscr{D}(\omega)$ is a low-rank structure, we propose an algorithm that applies rank reduction sequentially to each unfolded tensor:

`for` each frequency ω
$\tilde{\mathscr{D}}(\omega) = \mathscr{D}(\omega)$
`for` $n = 1,2,3$
unfold $\tilde{\mathscr{D}}(\omega)$ in mode-n $\rightarrow$ rank reduction (keep first r singular values)
refold to tensor $\tilde{\mathscr{D}}(\omega)$,
`end`
`end`
$\tilde{\mathscr{D}}(\omega)$ is the final rank-reduced tensor.

The above procedure can be used to denoise multidimensional seismic data in a similar way $f-x-y$ eigenimage filtering (Trickett, 2003) can be used to denoise data that depend on two spatial dimensions.

A reconstruction algorithm

The rank reduction iterative algorithm adopted by Oropeza and Sacchi (2011) and Kreimer and Sacchi (2011) for simultaneous denoising an reconstruction is given by

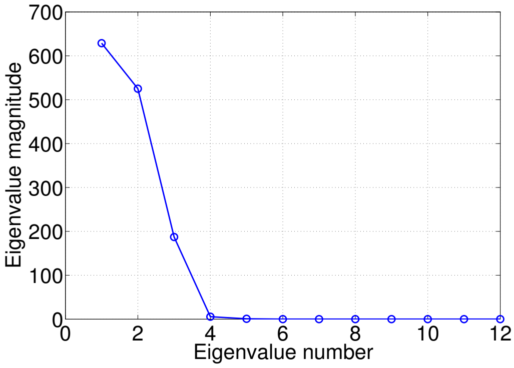
$$\mathscr{D}^k = a\,\mathscr{D}^{obs} + (\boldsymbol{1} - a\,\mathscr{T})\,\widetilde{\mathscr{D}}^{k-1}, \tag{1}$$

where a is a parameter between 0 and 1 that controls the level of reinsertion of noisy observations. The operator $\mathscr{T}$ is the sampling operator with the same dimensions of the data, filled with zeros in the bins with missing traces and ones in the bins containing samples. Notice that the reconstruction iterative algorithm in equation 1 could also use the HOSVD (Kreimer and Sacchi, 2011). We are surprised to notice that our simple rank reduction algorithm leads to results similar to those obtained using rank reduction iterative reconstruction via the HOSVD.

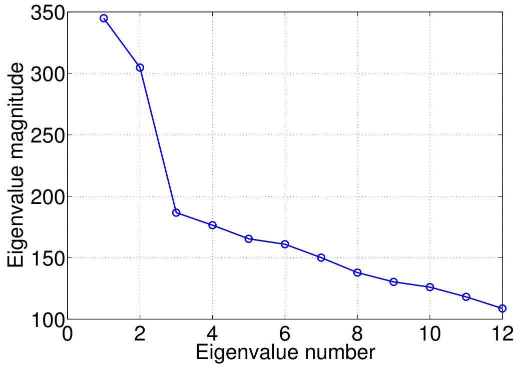
Examples

To demonstrate that unfolded tensors are a low rank structure we designed a volume of size $128 \times 12 \times 12 \times 12$ that contains 3 plane waves. Figure 1a displays the distribution of the eigenvalues for the unfolded tensor in mode-1. This figure corresponds to fully sampled data with $SNR = 100$. Clearly, only three singular values are different from zero. This coincides with the number of independent dips in the data. Figure 1b corresponds to the case where the data were contaminated with noise. The spectrum of singular values shows three dominant components immerse in smaller ones that model the noise subspace.

Figure 2 shows the result of applying the rank reduction procedure explained in the previous section.



(a) No missing traces and no noise



(b) 50% missing traces and $SNR = 1$

***Figure 1** : Distribution of eigenvalues for the mode-1 unfolding, for one frequency.*

We have kept the first 3 singular values in each unfolding. Column (a) in the figure is the non-decimated data (prior to noise contamination), the column (b) is the decimated data after addition of noise ($SNR = 1$). The latter is also the input to our algorithm. Column (c) is the reconstructed and denoised data and column (d) is the difference between the first and third column (error). The reconstructed and noise attenuated volume have negligible artifacts. Overall, one can confirm that the algorithm is able to recover the events with a high degree of accuracy. The reconstructed volume has a Frobenius norm that is about 9 times the Frobenius norm of the reconstruction error. As a comparison, we also used the truncated HOSVD. The results were very similar to those obtained with the proposed algorithm.

We also examine the spectrum of singular values of the unfolded tensor (mode 1) for a data set composed of three curved events. Figure 3 displays the distribution of eigenvalues for this exercise. The spectra of singular values for this example does not show an abrupt change in amplitude; a feature that can be used to reveal if the underlying data is a low rank structure. The reconstruction in this case is possible. However, the quality of reconstruction degrades with curvature. This is a result that was expected because the unfolded tensor is low rank when the data are composed of linear events.

Conclusions

We have presented a novel application of the singular value decomposition for rank reduction on a tensor. The algorithm operates directly on matrices obtained by unfolding the tensor. The proposed rank reduction method permits to denoise and reconstruct seismic data. This procedure, although different in nature, gives similar results to those obtained with the truncated HOSVD (Kreimer and Sacchi, 2011) .

Acknowledgements

The authors are grateful to the sponsors of the Signal Analysis and Imaging Group (SAIG) at the University of Alberta.

References

De Lathauwer, L., B. De Moor, and J. Vandewalle, 2000, A multilinear singular value decomposition: SIAM J. Matrix Anal. Appl., **21**, no. 4, 1253–1278.

Freire, S. and T. Ulrych, 1988, Application of singular value decomposition to vertical seismic profiling: Geophysics, **53**, 778–785.

Gao, J., M. D. Sacchi, and X. Chen, 2011, A fast rank reduction method for the reconstruction of 5D seismic volumes: SEG Technical Program Expanded Abstracts, **30**, 3622–3627.

Kreimer, N. and M. D. Sacchi, 2011, A tensor higher-order singular value decomposition (HOSVD) for pre-

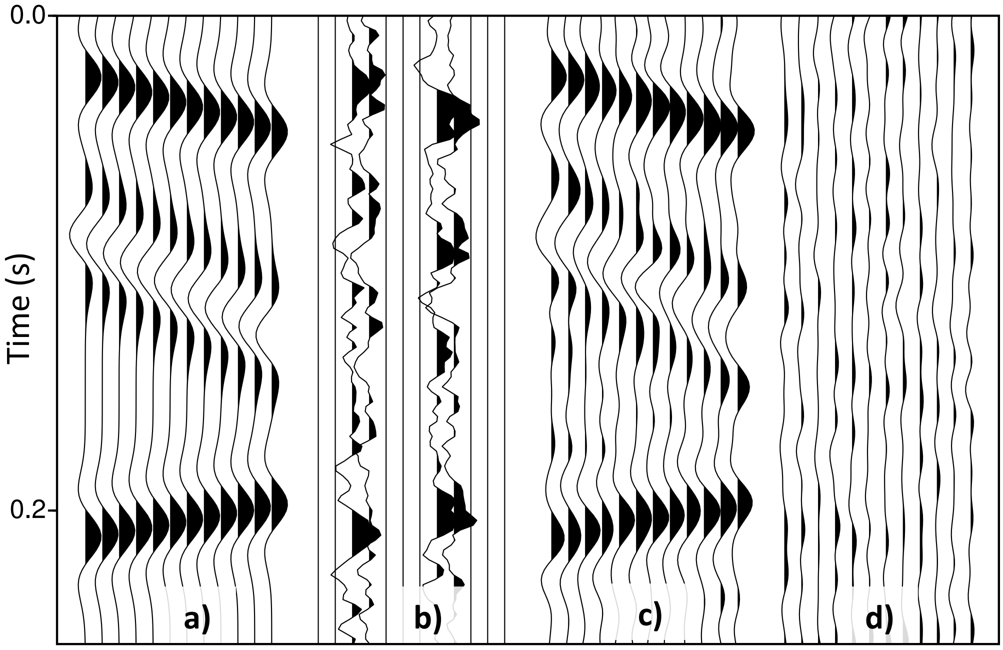


Figure 2 *: Reconstruction and noise attenuation for a 4D seismic volume with 3 linear events and* $SNR = 1$*. Only a subset of the volume is displayed in the figure.*

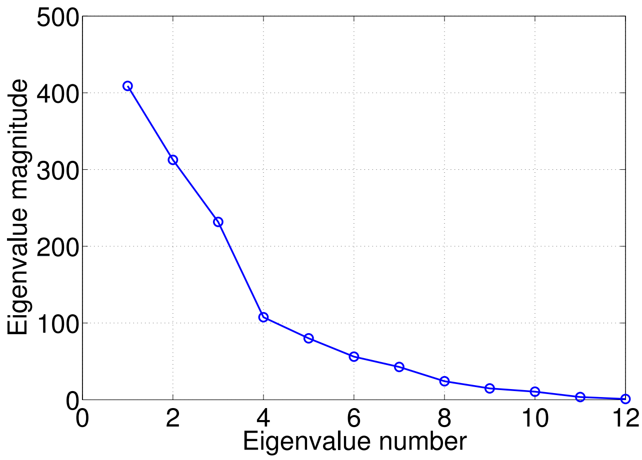


Figure 3 *: Distribution of eigenvalues for the mode-2 unfolding, for one frequency, curved events case. The volume is fully sampled and has no noise.*

stack simultaneous noise-reduction and interpolation: SEG Technical Program Expanded Abstracts, **30**, 3069–3074.
Oropeza, V. and M. Sacchi, 2011, Simultaneous seismic data denoising and reconstruction via multichannel singular spectrum analysis: Geophysics, **76**, V25–V32.
Trickett, S., 2008, *F-XY* Cadzow noise suppression: SEG, Expanded Abstracts, **27**, 2586–2590.
Trickett, S., L. Burroughs, A. Milton, L. Walton, and R. Dack, 2010, Rank-reduction-based trace interpolation: SEG, Expanded Abstracts, **29**, 3829–3833.
Trickett, S. R., 2003, *F-XY* eigenimage noise suppression: Geophysics, **68**, 751–759.