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New Approach to Finite-Difference Memory Variables by Using Lagrangian Mechanics

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Summary

In numerous algorithms for finite-difference modeling of seismic waves, the effects of attenuation are described by using memory variables. However, the physical meaning of memory variables and their limitations may be somewhat difficult to see, and they can be difficult to generalize for different types of attenuation, such as the one in poroelastic media. Here, we propose a different approach to memory variables, based on Lagrangian description of the rheological model called the Generalized Standard Linear Solid (GSLS). This approach allows treating the memory variables in a physically natural and interpretable way, and very similarly to the usual strain and stress. Such variables can therefore be readily generalized to other mechanisms of internal friction. Based on this approach, stress-strain relaxation laws for two types of deformation are derived. For a GSLS, the solutions are identical to those obtained by using the conventional approach.

Introduction

Numerical modeling is the key tool for extracting detailed information from seismic data, particularly in the presence of inelastic effects. Forward modeling is broadly utilized in seismic exploration, including acquisition system design, seismic migration, interpretation, and full waveform inversion. In the last of these applications, the accuracy of the modeling technique is critical. Although many techniques for seismic forward modeling have been developed, the finite-difference (FD) method is currently the most popular one, because of its ability to accurately model the seismic wave propagation in arbitrary heterogeneous media. The FD approach is especially suitable for modeling elastic seismic waves in the time domain.

In the existing viscoelastic approach, FD modeling of the inelastic effects presents some problems, because it requires knowledge of the entire time history of the material and evaluation of convolutional integrals in time. This problem was solved by Day and Minster (1984) and Carcione et al. (1988), who introduced additional variables, which are often called 'memory variables' now. Based on such memory variables, several FD codes, for 2-D and 3-D, viscoacoustic and viscoelastic seismic modeling were created (e.g., Robertsson et al., 1994; Bohlen, 2002). In all of these cases, the central question in FD modeling of energy dissipation is the construction of memory variables.

The memory variables by Day and Minster (1984) and Carcione et al. (1988) arise from postulating an anelastic stress-strain response of the material and approximating it by the Generalized Standard Linear Solid (GSLS). Such systems are often illustrated by combining multiple dashpots and springs (Figure 1). As shown below, in such an arrangement of mechanical elements, memory variables can be interpreted in a rather specific manner. Here, we propose a different approach to memory variables, which makes them closer to the first physical principles, more interpretable, and also much more general. We use Lagrangian mechanics to fully describe the model and produce all equations of motion. The new memory variables also allow straightforward extensions of the linear dissipation model to non-linear viscosity and other rigorous dissipation mechanisms.

Below, we briefly introduce the Lagrangian form of memory variables, illustrate their application to a single GSLS body, and compare them to Carcione's et al. (1988) approach. We also give two models of laboratory measurements of attenuation on a GSLS.

Theory

The GSLS model is generally considered suitable for explaining laboratory measurements of mechanical-energy dissipation in rock creep under stress, and it is used in many FD modeling algorithms (e.g., Carcione et al., 1988; Robertsson et al., 1994; Bohlen, 2002). Figure 1 shows a GSLS composed of N Maxwell bodies connected in parallel with a spring k. Note that from the viewpoint of Lagrangian mechanics of continuous media, such diagrams should be understood as representations of the structure of the Lagrangian. With no Maxwell bodies ($N = 0$), the Lagrangian represented by the single "k" symbol in Figure 1 is (Landau and Lifshitz, 1986):

$$L_0 = \frac{\rho}{2}\dot{u}_i\dot{u}_k - \left[\frac{\lambda}{2}\left(\varepsilon_{ii}\right)^2 + \mu\varepsilon_{ik}\varepsilon_{ik}\right], \tag{1}$$

where u_i is the displacement vector, ε_{ik} is the corresponding strain tensor (both indicated by the "external", observable variable e), ρ is the density, λ and μ are the Lamé moduli of the medium, and summations over all repeated indices are implied. With N Maxwell bodies added, if we denote their internal displacements by u_{Jk} and strains by ε_{Jik} (both shown by e_J in Figure 1), then the natural extension of the Lagrangian of the system is:

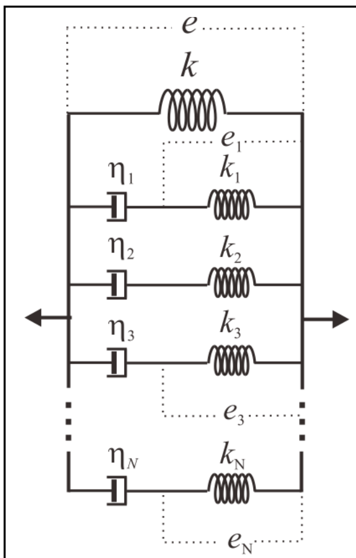

Figure 1.GSLS model. Parameters k_i are the spring constants and η_i are the corresponding viscosities of N Maxwell bodies connected in parallel with the main elastic spring k. The model can also be viewed as N Standard Linear Solids connected in parallel.

$$L = L_0 + \frac{1}{2}\sum_{J=1}^{N}\rho_J\dot{u}_{Jk}^2 - \sum_{J=1}^{N}\left[\frac{\lambda_J}{2}\left(\varepsilon_{Jik}\right)^2 + \mu\varepsilon_{Jik}\varepsilon_{Jik}\right], \tag{2}$$

where the subscript "J" is the counter of Maxwell bodies, and λ_J and μ_J are their Lamé parameters (shown as "springs" k_J in the diagram). In this expression, we also allow some densities, ρ_J, to be associated with the internal variables. Similarly, the internal friction depicted by the "dashpots" η_J in Figure 1 corresponds to the dissipation function (Landau and Lifshitz, 1986):

$$D = \sum_{J=1}^{N}\left[\frac{\eta_{J\lambda}}{2}\left(\dot{\varepsilon}_{ii} - \dot{\varepsilon}_{Jii}\right)^2 + \eta_{J\varepsilon}\left(\dot{\varepsilon}_{ik} - \dot{\varepsilon}_{Jik}\right)^2\right], \tag{3}$$

where $\eta_{J\lambda}$ and η_{ε} are the viscosity parameters for dilatation and shear associated with dissipation in Maxwell bodies.

Our approach to implementing a FD scheme for viscoelastic waveform modeling simply consists in using displacements u_{Jk} as the variables responsible for energy dissipation. The equations of motion for these variables are similar to those for the "external" displacement u_i, and follow from the Euler-Lagrange equations (Landau and Lifshitz, 1986):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_{Ji}}\right) - \frac{\partial L}{\partial q_{Ji}} = -\frac{\partial D}{\partial \dot{q}_{Ji}}, \tag{4}$$

where $q_{Ji} = u_i$ for $J = 0$ and $q_{Ji} = u_{Ji}$ for $J > 0$. These equations for the internal variables are very similar to those for the "external" field and can be readily implemented in the FD form.

Comparison to memory variables

The physical meaning of variables u_{Jk} as deformations of the internal springs is clear from the structure of the Lagrangian (Figure 1). The meaning of the traditional memory variables (Carcione et al., 1988) is somewhat more elaborate. Let us illustrate it on the example of a massless system ($\rho = 0, \rho_J = 0$) to which a time-dependent, spatially uniform and pure axial stress $\sigma(t)$ is applied. This model describes the typical creep or phase-lag attenuation testing of rock samples in the lab. The deformation of this system is described by a single "external" (measured) variable $e(t)$ and N internal variables $e_J(t)$ (internal strain of the springs in Maxwell bodies in Figure 1), and the (L, D) pair simplifies to:

$$\begin{cases} L = -\frac{1}{2}ke^2 - \sum_{J=1}^{N}\left(\frac{k_J}{2}e_J^2\right) + \sigma(t)e, \\ D = \sum_{J=1}^{N}\frac{\eta_J}{2}\left(\dot{e} - \dot{e}_J\right)^2, \end{cases} \tag{5}$$

where $J = 1, 2, ..., N$, and the external force term $\sigma(t)e$ is added to the Lagrangian (Landau and Lifshitz, 1976). The equations of motion (4) for this system become:

$$\begin{cases} ke - \sigma(t) = -\sum_{J=1}^{N}\eta_J\left(\dot{e} - \dot{e}_J\right), \\ k_J e_J = \eta_J\left(\dot{e} - \dot{e}_J\right), \end{cases} \tag{6}$$

with strains at time $t = 0$ equal $e_0 = e(0) = e_J(0) = \sigma(0)/k_U = \sigma_0/k_U$, where $k_U = k + \sum_{J=1}^{N} k_J$ is the "unrelaxed" spring constant.

From the second eq. (5), the relationship between the external strain and the internal strain can be written in an integral form:

$$e_J(t) = \int_{-\infty}^{t}\dot{e}(\tau)\exp\left(-\frac{1}{\tau_J}(t-\tau)\right)d\tau, \tag{7}$$

where $\tau_J = \eta_J/k_J$ is the relaxation time for the J^{th} Maxwell body.

Eq. (7) is a time convolution which can be viewed as a retarded response of strain $e_J(t)$ to variable $e(t)$. Note that such relation is only possible because the internal variables in the GSLS are taken as massless. Further, from the first eq. (5), the stress-strain relation can be described in terms of both the "relaxed" (k) and unrelaxed spring constants (k_U):

$$\sigma(t) = ke + \sum_{J=1}^{N}\eta_J\left(\dot{e} - \dot{e}_J\right) = k_U e - \sum_{J=1}^{N}\left[k_J e - \eta_J\left(\dot{e} - \dot{e}_J\right)\right] = k_U e - \sum_{J=1}^{N}k_J\left(e - e_J\right). \tag{8}$$

Comparing the last of these relations to the similar relation using memory variables (eq. (26) in Carcione et al., 1988) shows that the J^{th} memory variable in this case equals: $p_J(t) = -k_J\left(e - e_J\right)$. Thus, the conventional memory variables can be interpreted as the reductions of stresses produced by the internal springs, resulting from the deformations of the dashpots. By comparison, our internal variables are simply the deformations e_J, which are (in principle) measurable, can be understood similar to the external deformation e and obey similar equations of motion. In addition, the use of memory variables $p_J(t)$ implies the knowledge of the "unrelaxed" modulus k_U, whereas the Lagrangian variables e_J interact with the usual elastic modulus k.

Examples

Let us illustrate the behaviour of FD internal variables e_J on the deformation of a single GSLS body. Figure 2 shows the quality factor for a GSLS containing five Maxwell bodies (Table 1), as a function of frequency (O'Connell and Budiansky, 1978). The parameters of the body are selected so that $Q \approx 100$ within the frequency band from 0.01 Hz to 100 Hz. Note that the number of Maxwell bodies can be counted by the peaks in Q^{-1} shown in this Figure.

To understand the behavior of the memory variables, we consider two different cases similar to creep testing of rock samples in the lab. First, under a step in stress, $\sigma(t) = \sigma_0 H(t)$ (where $H(t)$ is the Heaviside function), the "external" strain shows an instantaneous increase e_0 followed by an exponential creep (Figure 3a). The empirical modulus of the GSLS system, $M(t) = \sigma(t)/e(t)$, decreases with time

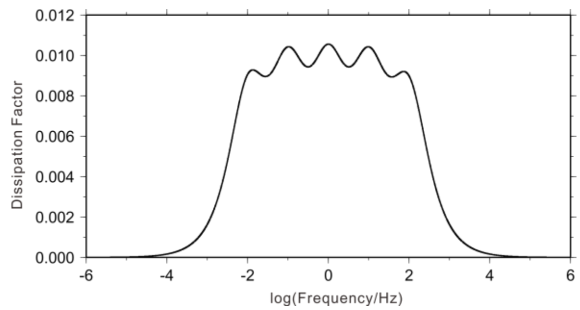

Figure 1. Dissipation factor as a function of frequency for a GSLS model with five Maxwell bodies.

Table 1. Parameters of Maxwell bodies in the GSLS model

J	k_J (MPa)	η_J (Pa·s)
k	1000	
1	15	$9.3 \cdot 10^9$
2	15	$9.3 \cdot 10^8$
3	15	$9.3 \cdot 10^7$
4	15	$9.3 \cdot 10^6$
5	15	$9.3 \cdot 10^5$

from unrelaxed $M_U = \sigma_0/e_0$ to relaxed $M_R = \sigma_0/e(t \to \infty)$ .These effects occurs because of the internal strains decreasing from e_0 to zero (Figure 3b). Because of the different viscosities, the strain rates for these five Maxwell bodies are very different. Note that the lowest-viscosity Maxwell bodies are less separable in such deformation (nearly overlapping purple, and blue lines in Figure 3b), and the time for the external strain almost equals that of the Maxwell body with the highest viscosity (black line in Figure 3b).

Similar conclusions arise from testing by a step in strain ($e(t) = e_0 H(t)$; Figure 4).The empirical modulus of the system with constant strain shows a similar (somewhat faster) decrease from the level of M_U to M_R with time (Figure 4a). However, the five Maxwell bodies deform independently in this case (Figure 4b). Thus, both constant-stress and constant-strain relaxations occur through a decrease of the internal strains. This leads to a decrease of the internal energy $E(t) = \frac{1}{2}\sum_{J=1}^{N} k_J e_J^2(t)$.

Conclusions

We showed how the memory variables used to represent anelastic responses in finite-difference (FD) modeling can be obtained from Lagrangian mechanics. The Lagrangian description leads to new "internal strain" variables, which are simpler and physically more meaningful than the conventional memory variables and allow generalization to any types of internal friction. This description is also accurate, and allows a straightforward FD implementation. Examples of constant-strain and constant-stress deformations show that the relaxation occurs through the decrease of the internal strains with time.

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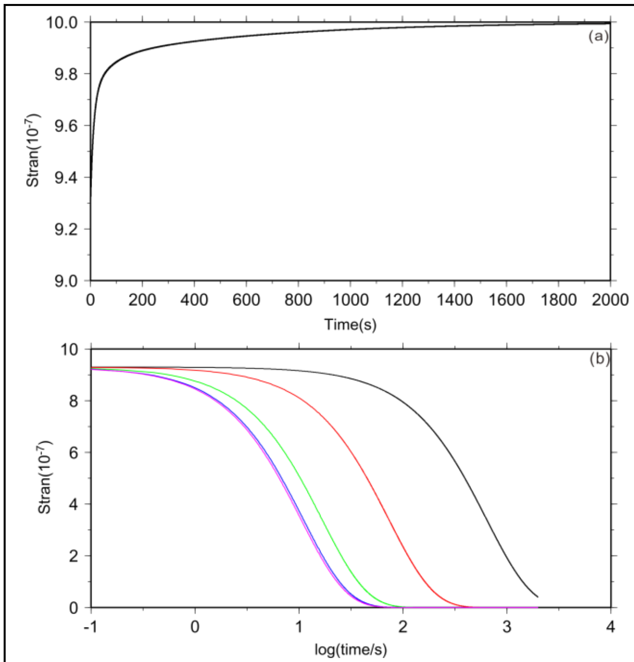

Figure 3

Relaxation of a GSLS under constant stress: a) the external strain; b) the strains of the internal springs. The five lines in (b) indicate the five Maxwell bodies, with viscosities increasing from purple to black colours.

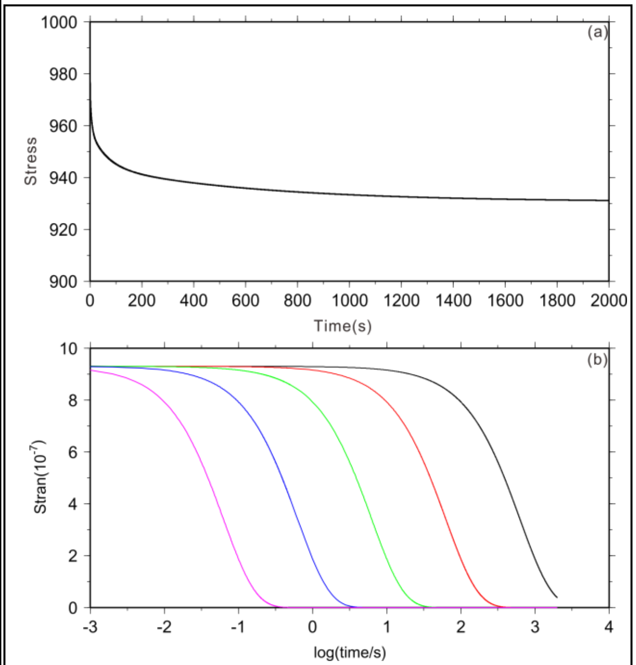

Figure 4

Relaxation of a GSLS under constant strain: a) Variation of total stress; b) the strains in the internal springs. Inline colours in (b) denote the five Maxwell bodies, as in Figure 3.